1. Let $S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ denote the unit sphere in $\mathbb{R}^3$. Evaluate the surface integral over $S$:
$$\int_S (x^2 + y + z) dA.$$

2. Evaluate the line integral
$$\int_C (2x - y) dx + (x + 3y) dy$$
along the path $C$: elliptic path $x = 4 \sin t, y = 3 \cos t$ from $(0, 3)$ to $(4, 0)$.

3. Use Green’s theorem to evaluate the line integral
$$\int_C (x^2 - y^2) dx + 2xy dy$$
along the path $C : x^2 + y^2 = 16$.

4. Let $F$ be a field. For $m$ and $n$ positive integers, let $M_{m \times n}$ be the vector space of $m \times n$ matrices over $F$. Fix $m$ and $n$, and fix matrices $A$ and $B$ in $M_{m \times n}$. Define the linear transformation $T$ from $M_{m \times n}$ to $M_{m \times n}$ by
$$T(X) = AXB.$$

Prove that if $m \neq n$, then $T$ is not invertible.

5. Let $T$ be a real, symmetric, $n \times n$ tridiagonal matrix:
$$T = \begin{bmatrix}
a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 \\
b_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 \\
0 & b_2 & a_3 & b_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\
0 & 0 & 0 & \cdots & b_{n-1} & a_n \\
\end{bmatrix}$$
(All entries not on the main diagonal or the diagonal just above and below the main one are zero.) Assume \( b_j \neq 0 \) for all \( j \). Prove
(a) \( \text{rank}(T) \geq n - 1 \).
(b) \( T \) has \( n \) distinct eigenvalues.

6. Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear transformation, where \( n > 1 \). Prove that there is a 2-dimensional subspace \( M \subseteq \mathbb{R}^n \) such that \( T(M) \subseteq M \).

7. Let \( E \) be a three-dimensional vector space over \( \mathbb{Q} \). Suppose \( T : E \to E \) is a linear transformation and \( Tx = y, Ty = z, Tz = x + y \), for certain \( x, y, z \in E, x \neq 0 \). Prove that \( x, y, \) and \( z \) are linearly independent.

8. Show that the system of differential equations
\[
\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]
has a solution which tends to \( \infty \) as \( t \to -\infty \) and tends to the origin as \( t \to \infty \).

9. Let the function \( x(t)(-\infty < t < \infty) \) be a solution of the differential equation
\[
\frac{d^2 x}{dt^2} - 2b \frac{dx}{dt} + cx = 0
\]
such that \( x(0) = x(1) = 0 \). (Here \( b \) and \( c \) are real constants.) Prove that \( x(n) = 0 \) for every integer \( n \).

10. Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded continuously differentiable function. Show that every solution of \( y'(x) = f(y(x)) \) is monotone, i.e., either monotonically increasing or decreasing.

11. Determine the last decimal digit of \( A = 17^{17^{17}} \).

12. Determine the greatest common divisor of the elements of the set \( \{ n^{13} - n | n \in \mathbb{Z} \} \).

13. Let the function \( f : [0, 1] \to [0, 1] \) have the following properties:
(a) \( f \) is of class \( C^1 \).
(b) \( f(0) = f(1) = 0 \).
(c) \( f' \) is nonincreasing.

Prove that the arclength of the graph of \( f \) does not exceed 3.

14. Show that a positive constant \( t \) can satisfy
\[
e^x > x^t \quad \forall x > 0
\]
if and only if \( t < e \).
15. Prove that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ which maps open sets to open sets must be monotonic.

16. The Fibonacci numbers $f_1, f_2, \cdots$ are defined recursively by $f_1 = 1, f_2 = 2$, and $f_{n+1} = f_n + f_{n-1}$, $\forall n \geq 2$. Show that

$$\lim_{n \to \infty} \frac{f_{n+1}}{f_n}$$

exists and evaluate the limit.

17. Let $G$ be a finite group, with identity $e$. Suppose that for every $a, b \in G$ distinct from $e$, there is an automorphism $\sigma$ of $G$ such that $\sigma(a) = b$. Prove that $G$ is abelian.

18. Prove that the group of automorphisms of a cyclic group of prime order $p$ is cyclic and find its order.

19. Let $G$ be a group of order 10 which has a normal subgroup of order 2. Prove that $G$ is abelian.

20. Prove that $\mathbb{Q}$, the additive group of rational numbers, can not be written as the direct sum of two nontrivial subgroups.