Section 3.4 - Proofs Involving Conjunctions and Biconditionals

3.4.3. Suppose $A \subseteq B$. Prove that for every set $C$, $C \setminus B \subseteq C \setminus A$.

**Givens**

$A \subseteq B$

**Goal**

$C \setminus B \subseteq C \setminus A$

Expanding using the definition of subset gives

**Givens**

$\forall x (x \in A \rightarrow x \in B)$

**Goal**

$\forall x (x \in (C \setminus B) \rightarrow x \in (C \setminus A))$

We can now let $x$ be arbitrary in both the given and goal and bring the left hand side of the goal to the givens

**Givens**

$x \in A \rightarrow x \in B$

$x \in C \setminus B$

**Goal**

$x \in C \setminus A$

Finally by set difference we have

**Givens**

$x \in A \rightarrow x \in B$

$x \in C \setminus B$

**Goal**

$x \in C \land x \notin A$

Separating the **and** statements gives

**Givens**

$x \in A \rightarrow x \in B$

$x \in C$

$x \notin B$

**Goal**

$x \in C$

$x \notin A$

Thus we will prove each goal separately. First, since $x \in C$ the first goal is clearly satisfied. Second, since $x \notin B$ by modus tollens since $x \in A \rightarrow x \in B$ gives $x \notin A$ which proves the second goal.

Formally,

**Theorem.** Suppose $A \subseteq B$. Then for every set $C$, $C \setminus B \subseteq C \setminus A$.

**Proof.** Let $x$ be an arbitrary element with $x \in C \setminus B$. Then $x \in C$ and $x \notin B$. Since $x \notin B$ and $A \subseteq B$ we know that $x \notin A$. But then since $x \in C$ and $x \notin A$ we can conclude that $x \in C \setminus A$ and because $x$ was arbitrary that $C \setminus B \subseteq C \setminus A$. 

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3.4.17. Suppose $F$ and $G$ are families of sets.

a. Prove that $\bigcup(F \cap G) \subseteq (\bigcup F) \cap (\bigcup G)$.

**Given:** $F$ and $G$ are families

**Goal:** $\bigcup(F \cap G) \subseteq (\bigcup F) \cap (\bigcup G)$

Expanding using subset and letting $x$ be arbitrary gives

**Given:** $F$ and $G$ are families

$x \in \bigcup(F \cap G)$

Which by the definition of intersection and family union gives

**Given:** $F$ and $G$ are families

$\exists A \in (F \cap G)(x \in A)$

Letting $A = A_0$ in the given and separating the goal gives

**Given:** $F$ and $G$ are families

$(A_0 \in (F \cap G)) \land (x \in A_0)$

Finally applying the definition of intersection and separating the given gives

**Given:** $F$ and $G$ are families

$A_0 \in F \land A_0 \in G$

$x \in A_0$

Now we can easily prove the first goal since $x \in A_0$ and $A_0 \in F$ so letting $B = A_0$ implies that $\exists B \in F(x \in B)$ or $x \in \bigcup F$. Similarly since $x \in A_0$ and $A_0 \in G$ letting $C = A_0$ we know that $\exists C \in G(x \in C)$ or $x \in \bigcup G$. Since $x \in \bigcup F$ and $x \in \bigcup G$ we know $x \in (\bigcup F) \cap (\bigcup G)$.

Formally,

**Theorem.** Suppose $F$ and $G$ are families of sets. Then $\bigcup(F \cap G) \subseteq (\bigcup F) \cap (\bigcup G)$.

**Proof.** Let $x$ be an arbitrary element with $x \in \bigcup(F \cap G)$. Since $x \in \bigcup(F \cap G)$ let $A_0$ be a set such that $x \in A_0$ and $A_0 \in F \cap G$. Then $A_0 \in F$ and $A_0 \in G$. Since $x \in A_0$ and $A_0 \in F$ we have that $\exists A \in F(x \in A)$ which gives $x \in \bigcup F$. Similarly since $x \in A_0$ and $A_0 \in G$ we have that $\exists B \in G(x \in B)$ which gives $x \in \bigcup G$. Because $x \in \bigcup F$ and $x \in \bigcup G$ we can conclude that $x \in (\bigcup F) \cap (\bigcup G)$ and since $x$ is arbitrary that $\bigcup(F \cap G) \subseteq (\bigcup F) \cap (\bigcup G)$.
b. What is wrong with the following proof that \((\cup F) \cap (\cup G) \subseteq \cup (F \cap G)\).

Proof. Suppose \(x \in (\cup F) \cap (\cup G)\). This means that \(x \in \cup F\) and \(x \in \cup G\), so \(\exists A \in F(x \in A)\) and \(\exists A \in G(x \in A)\). Thus, we can choose a set \(A\) such that \(A \in F\), \(A \in G\), and \(x \in A\). Since \(A \in F\) and \(A \in G\), \(A \in F \cap G\). Therefore \(\exists A \in F \cap G(x \in A)\), so \(x \in \cup (F \cap G)\). Since \(x\) was arbitrary, we can conclude that \((\cup F) \cap (\cup G) \subseteq \cup (F \cap G)\).

The problem with this proof is in the selection of the same existential instantiation \(A\) for \(\exists A \in F(x \in A)\) and \(\exists A \in G(x \in A)\). The existence statements mean there are sets for each family, but they are not necessarily the same sets. Thus if the element came from different sets in \(F\) and \(G\) then those sets would not be in \(F \cap G\).

c. Consider the families \(F = \{\{1\}\}\) and \(G = \{\{1, 2\}\}\). Then \(\cup F = \{1\}\), \(\cup G = \{1, 2\}\) and hence \((\cup F) \cap (\cup G) = \{1\}\). But clearly \(F \cap G = \emptyset\) since \(F\) and \(G\) contain no sets in common. This gives \(\cup (F \cap G) = \emptyset \neq \{1\}\).

Section 3.5 - Proofs Involving Disjunctions

3.5.2. Suppose \(A\), \(B\), and \(C\) are sets. Prove that \((A \cup B) \setminus C \subseteq A \cup (B \setminus C)\).

<table>
<thead>
<tr>
<th>Givens</th>
<th>Goal</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A), (B), and (C) are sets</td>
<td>((A \cup B) \setminus C \subseteq A \cup (B \setminus C))</td>
</tr>
</tbody>
</table>

Applying the definition of subset to the goal gives

<table>
<thead>
<tr>
<th>Givens</th>
<th>Goal</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A), (B), and (C) are sets</td>
<td>(\forall x(x \in (A \cup B) \setminus C \rightarrow x \in A \cup (B \setminus C)))</td>
</tr>
</tbody>
</table>
Given Goal

\[
A, B, \text{ and } C \text{ are sets} \\
x \in A \lor x \in (B \setminus C) \\
x \notin C
\]

Now we will consider the two cases from the disjunction in the given:

Case 1: Suppose \( x \in A \). Then clearly the first part of the goal is satisfied (and \( x \in A \cup (B \setminus C) \)).

Case 2: Suppose \( x \in B \). Then since \( x \notin C \) gives \( x \in B \setminus C \) which is the second part of the goal (and again \( x \in A \cup (B \setminus C) \)).

Formally,

**Theorem.** Suppose \( A, B, \text{ and } C \) are sets. Then \( (A \cup B) \setminus C \subseteq A \cup (B \setminus C) \).

**Proof.** Suppose that \( x \) is arbitrary and \( x \in (A \cup B) \setminus C \). Then \( x \in A \cup B \) and \( x \notin C \). Since \( x \in A \cup B \) we know that either \( x \in A \) or \( x \in B \) so consider two cases:

Case 1: Suppose \( x \in A \). Then clearly \( x \in A \cup (B \setminus C) \).

Case 2: Suppose \( x \in B \). Then since \( x \notin C \) we have that \( x \in B \setminus C \) which is the second part of the goal (and again \( x \in A \cup (B \setminus C) \)).

Thus in either case \( x \in A \cup (B \setminus C) \) and since \( x \) was arbitrary that \( (A \cup B) \setminus C \subseteq A \cup (B \setminus C) \).

3.5.27. Consider the following putative theorem.

**Theorem?** For any sets \( A, B, \text{ and } C \), if \( A \setminus B \subseteq C \) and \( A \nsubseteq C \) then \( A \cap B \neq \emptyset \).

Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

**Proof.** Since \( A \nsubseteq C \), we can choose some \( x \) such that \( x \in A \) and \( x \notin C \). Since \( x \notin C \) and \( A \setminus B \subseteq C \), \( x \notin A \setminus B \). Therefore either \( x \notin A \) or \( x \in B \). But we already know that \( x \in A \), so it follows that \( x \in B \). Since \( x \in A \) and \( x \in B \), \( x \in A \cap B \). Therefore \( A \cap B \neq \emptyset \).

The proof (and thus the theorem) is correct. First since \( A \nsubseteq C \) means \( \exists x (x \in A \land x \notin C) \) and thus it employs existential instantiation to assert that an \( x \) can be chosen (although using \( x_0 \) may have been a better choice of symbol). Then it uses a modus tollens argument since \( x \notin C \) but \( A \setminus B \subseteq C \) (i.e. \( \forall x (x \in A \setminus B \rightarrow x \in C) \) to assert that \( x \notin A \setminus B \). Using the definition of set difference and applying DeMorgan’s law then asserts that \( x \notin A \lor x \in B \). We can then use a variant of modus ponens (since \( x \notin A \lor x \in B \Rightarrow x \in A \rightarrow x \in B \) and \( x \in A \)) to assert that \( x \in B \). Finally using the definition of intersection gives that \( x \in A \) and \( x \in B \) means \( x \in A \cap B \) and therefore \( A \cap B \neq \emptyset \).
Section 3.6 - Existence and Uniqueness Proofs

3.6.6. Let $U$ be any set.

a. Prove that there is a unique $A \in \mathcal{P}(U)$ such that for every $B \in \mathcal{P}(U)$, $A \cup B = B$.

Existence. Let $A = \emptyset$. Clearly since $A \in \mathcal{P}(U)$ (meaning $A \subseteq U$), for $A = \emptyset$ gives $\emptyset \subseteq U$ which is true for all $U$. Furthermore, for every set $B \in \mathcal{P}(U)$, $A \cup B = \emptyset \cup B = B$.

Uniqueness. Now we must show that for any other set $C$ that $C = \emptyset$. Assume $C \in \mathcal{P}(U)$ such that for all $B \in \mathcal{P}(U)$ that $C \cup B = B$. Let $B = \emptyset$ which clearly is $\in \mathcal{P}(U)$. This implies that $C \cup \emptyset = \emptyset$ but we know that $C \cup \emptyset = C$. Thus $C = \emptyset$.

b. Prove that there is a unique $A \in \mathcal{P}(U)$ such that for every $B \in \mathcal{P}(U)$, $A \cup B = A$.

Existence. Let $A = U$. Clearly since $A \in \mathcal{P}(U)$ (meaning $A \subseteq U$), for $A = U$ gives $U \subseteq U$ which is true for all $U$. Furthermore, for every set $B \in \mathcal{P}(U)$ (meaning $B \subseteq U$) giving $\forall x (x \in B \rightarrow x \in U)$. Therefore $A \cup B = U \cup B = U$.

Uniqueness. Now we must show that for any other set $C$ that $C = U$. Assume $C \in \mathcal{P}(U)$ such that for all $B \in \mathcal{P}(U)$ that $C \cup B = B$. Let $B = U$ which clearly is $\in \mathcal{P}(U)$. This implies that $C \cup U = C$ but we know that $C \cup U = U$. Thus $C = U$.

3.6.10. Suppose $A$ is a set, and for every family of sets $\mathcal{F}$, if $\cup \mathcal{F} = A$ then $A \in \mathcal{F}$. Prove that $A$ has exactly one element.

We will tackle this proof through contradiction by assuming that $A$ does not have exactly one element. This means that it either has no elements (existence) or multiple elements (uniqueness).

Existence. Assume $A = \emptyset$ and $\cup \mathcal{F} = \emptyset$. Then since $\cup \mathcal{F} = A$ gives that $A \in \mathcal{F}$, i.e. $\emptyset \in \mathcal{F}$, and hence $\mathcal{F} \neq \emptyset$. However if $\cup \mathcal{F} = \emptyset$ then $\mathcal{F} = \emptyset$ which is a contradiction and thus $A$ must have at least one element.

Uniqueness. Assume $A$ has more than one element, specifically let $A = \{x, y\}$ giving $x \in A$ and $y \in A$ with $x \neq y$. Furthermore, let $\mathcal{F} = \{\{x\}, \{y\}\}$. Clearly then $\cup \mathcal{F} = A$ so $A \in \mathcal{F}$. Yet $A \notin \mathcal{F}$ which is a contradiction and hence $A$ cannot have more than one element.

Therefore if $A$ must have at least one element but $A$ cannot have more than one element, then $A$ must have exactly one element.