Double shuffle relations of double zeta values and the double Eisenstein series at level $N$

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Abstract
In their seminal paper, Gangl, Kaneko and Zagier defined a double Eisenstein series and used it to study the relations between double zeta values. One of their key ideas is to study the formal double space and apply the double shuffle relations. They also proved the double shuffle relations for the double Eisenstein series. More recently, Kaneko and Tasaka extended the double Eisenstein series to level 2, proved its double shuffle relations and studied the double zeta values at level 2. Motivated by the above works, we define in this paper the corresponding objects at higher levels and prove that the double Eisenstein series at level $N$ satisfies the double shuffle relations for every positive integer $N$. In order to obtain our main theorem, we prove a key result on the multiple divisor functions at level $N$ and then use it to solve a complicated under-determined system of linear equations by some standard techniques from linear algebra.

1. Introduction
Eisenstein series have played important roles in the study of modular forms and elliptic curves. One of their most important properties is that the constant term of their Fourier series expansion is essentially given by the Riemann zeta values at even weight. In the seminal paper [7], Gangl, Kaneko and Zagier defined a double Eisenstein series and used it to study the relations between double zeta values. One of their key ideas is to study the formal double space and apply the double shuffle relations. They then proved the double shuffle relations for the double Eisenstein series. The double zeta relations have also been considered by Baumard and Schneps [5] from the point of view of period polynomials and the double shuffle Lie algebra defined by Ihara. More recently, Kaneko and Tasaka [9] and Nakamura and Tasaka [11] extended the double Eisenstein series to level 2, proved its double shuffle relations and studied the double zeta values at level 2. Motivated by the above works, we define in this paper the corresponding objects at higher levels and consider the double shuffle relations satisfied by them.

We follow the notation in [7]: for any $m, c \in \mathbb{Z}$ and $\tau \in \mathbb{H}$ (upper half-plane), we write $m\tau + c > 0$ if $m > 0$ or $m = 0$ and $c > 0$ and $m\tau + c > n\tau + d$ if $m\tau + c - (n\tau + d) > 0$. For any $a = (a_1, \ldots, a_d) \in (\mathbb{Z}/N\mathbb{Z})^d$ and $s = (s_1, \ldots, s_d) \in \mathbb{N}^d$ we define the multiple Eisenstein series at level $N$ by

$$G_{a}^{s} (\tau) = G_{a}^{s;N} (\tau) = \sum_{\substack{m_1 N\tau + c_1 \succ \cdots \succ m_d N\tau + c_d \succ 0 \\text{ and } \\forall j \ \ m_j, c_j \in \mathbb{Z}, \ c_j \equiv a_j (\text{mod } N) \ }} \frac{1}{(m_1 N\tau + c_1)^{s_1} \cdots (m_d N\tau + c_d)^{s_d}}.$$ (1.1)

Here, by convention, we often choose $0 \leq a < N$ to represent the residue class congruent to $a$ modulo $N$. It is not too hard to show that the series converges absolutely when $s_1 \geq 3$ and $s_j \geq 2$ for all $j \geq 2$. We will call $d$ the depth and the sum $s_1 + \cdots + s_d$ the weight. In the level one case, Gangl, Kaneko and Zagier [7] studied the double Eisenstein series and related them...
to modular form by using the Eichler–Shimura correspondence. In [3], Bachmann generalized this to arbitrary depth and obtained many interesting relations among these and the classical Eisenstein series (and the cusp form Δ) using the double shuffle relations.

The main idea to study the multiple Eisenstein series is to use their Fourier series expansions with the help of so-called multiple divisor functions at level $N$ defined as follows: For $\mathbf{a} = (a_1, \ldots, a_d) \in (\mathbb{Z}/\mathbb{N})^d$ and $\mathbf{s} = (s_1, \ldots, s_d) \in (\mathbb{N} \cup \{0\})^d$, $\sigma_{\mathbf{s}}^{\mathbf{a}}(m) = \sigma_{\mathbf{s}}^{\mathbf{a}, N}(m) = \sum_{u_1 v_1 + \cdots + u_d v_d = m; \ u_j, v_j \in \mathbb{N} \ \forall j} \eta^{a_1 v_1 + \cdots + a_d v_d} u_1^{s_1} \cdots v_d^{s_d}$, \hspace{1cm} (1.2)

where $\eta = \eta_N = \exp(2\pi i/N)$ is the primitive $N$th root of unity and $v_1, \ldots, v_d$ are positive integers. Obviously, one can recover the classical divisor function by setting $N = d = 1$.

We now briefly describe the content of the paper. In the next section, we shall first define the multiple zeta values at level $N$ and write down explicitly the double shuffle relations satisfied by these values. Then we consider the same problem in the formal vector space corresponding to arbitrary depth and obtained many interesting relations among these and the classical modular form by using the Eichler–Shimura correspondence. In [3], Bachmann generalized this to arbitrary levels lies in the fact that there are many choices of the constant terms in the generating function of the double Eisenstein series and other related series. It turns out that this fact is a consequence of an under-determined system of linear equations with $(3N^2 + 1)/2$ variables and $N^2 + N$ equations. Essentially, we need to know whether the corresponding level-$N$ multiple Eisenstein series also satisfy similar relations. When $N = 1$, this has been studied by Bachmann and Tasaka [4].

The main goal of the paper is to prove Theorem 6.5, which gives the double shuffle relations of the double Eisenstein series at level $N$ for every positive integer $N$. The difficulty in generalizing the known $N = 1$ and $N = 2$ cases to arbitrary levels lies in the fact that there are many choices of the constant terms in the generating function of the double Eisenstein series and other related series. It turns out that this fact is a consequence of an under-determined system of linear equations with $(3N^2 + 1)/2$ variables and $N^2 + N$ equations. Essentially, we need to know whether these equations are consistent with each other. For this, we need a key result concerning $N$th roots of unity and the multiple divisor functions at level $N$ which will be proved in Section 7. In the last section, using some standard techniques from linear algebra, we prove the solvability of the linear system mentioned above. This enables us to derive our main result, Theorem 6.3, which generalizes [7, Theorem 7; 9, Theorem 3].

2. Multiple zeta values at level $N$

For any $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d$ with $s_1 \geq 2$ and $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}/\mathbb{N}^d$, we define the multiple zeta values at level $N$ by

$$\zeta_{\mathbf{s}}^{\mathbf{a}}(N) = \sum_{n_1 > \cdots > n_d > 0, \ n_j \equiv a_j \ (\text{mod} \ N) \ \forall j} \frac{1}{n_1^{s_1} \cdots n_d^{s_d}}. \hspace{1cm} (2.1)$$

These numbers are rational multiples of the multiple Hurwitz zeta values which have been studied by many authors. If we define the multiple polylogarithms

$$\operatorname{Li}_{\mathbf{s}}(x_1, \ldots, x_d) = \sum_{n_1 > \cdots > n_d > 0} \frac{x_1^{s_1} \cdots x_d^{s_d}}{n_1^{s_1} \cdots n_d^{s_d}},$$

then it is not hard to see that

$$\zeta_{\mathbf{s}}^{\mathbf{a}}(N) = \frac{1}{N^d} \sum_{\beta_1 = 1}^{N} \ldots \sum_{\beta_d = 1}^{N} \eta^{-\beta_1 a_1 - \cdots - \beta_d a_d} \operatorname{Li}_{\mathbf{s}}(\eta^{\beta_1}, \ldots, \eta^{\beta_d}). \hspace{1cm} (2.2)$$
We now restrict ourselves to levels 1 or 2. We remark that our definition of the double zeta value at level 2 is slightly different from that of [9] since we allow \( a_1 = a_2 = 0 \), in which case we in fact essentially recover the usual double zeta values (at level 1).

We can also use Chen’s iterated integrals to derive formulas similar to (2.2) which will be useful in the regularization of these values. Let \( \omega = dt/t \), \( \omega^a_{\alpha} = \omega(N)_{\alpha}^a = (\eta^a t)^{\alpha - 1} dt/(1 - \eta^a t) \) (\( 1 \leq a \leq N \)) be 1-forms. By the partial fraction expansion

\[
\frac{t^{a-1}}{1 - t^N} = \frac{1}{N} \sum_{\alpha=1}^{N} \eta^{-\alpha(a-1)} \frac{1}{1 - \eta^a t}
\]

we see immediately that

\[
\zeta_N^a(r) = \int_0^1 \omega^{-1} t^{a-1} dt \left( \frac{\omega}{1 - t^N} \right) = \frac{1}{N} \sum_{\alpha=1}^{N} \omega^{-1} \frac{\eta^{-\alpha(a-1)}}{1 - \eta^a t} = \frac{1}{N} \sum_{\alpha=1}^{N} \eta^{-\alpha a} L_i, \eta^a. \tag{2.3}
\]

Furthermore,

\[
\zeta_N^{a,b}(r, s) = \int_0^1 \omega^{-1} t^{a-1} dt \left( \frac{\omega_s^{b-1}}{1 - t^N} \right) = \frac{1}{N^2} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \omega^{-1} \frac{\eta^{-\alpha(a-b-1)}}{1 - \eta^a t} \frac{\eta^{-\beta(b-1)}}{1 - \eta^\beta t}
\]

\[
= \frac{1}{N^2} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \eta^{-\alpha(a-b) - \beta b} L_{i_r, s} (\eta^a, \eta^b - \alpha). \tag{2.4}
\]

Our first result is the following explicit form of double shuffle relations.

**Proposition 2.1.** For positive integers \( r, s \geq 2 \) and integers \( a, b \in \mathbb{Z}/N\mathbb{Z} \), we have

\[
\zeta_N^a(r) \zeta_N^b(s) = \zeta_N^a(r, s) + \zeta_N^b(s, r) + \delta_{a, b} \zeta_N^a(s + r)
\]

\[
= \sum_{i=0}^{s-1} \binom{r + i - 1}{r - 1} \zeta_N^{a+b}(r + i, s - i) + \sum_{j=0}^{r-1} \binom{s + j - 1}{s - 1} \zeta_N^{a+b}(s + j, r - j)
\]

\[
= \sum_{i+j=r+s} \binom{i-1}{r-1} \zeta_N^{a+b}(i, j) + \binom{i-1}{s-1} \zeta_N^{a+b}(i, j), \tag{2.5}
\]

where \( \delta_{a, b} \) is the Kronecker symbol, namely \( \delta_{a, b} = 1 \) if \( a = b \) and \( \delta_{a, b} = 0 \) if \( a \neq b \).

**Proof.** The first equality is clear by definition (2.1). The second equality follows immediately from the shuffle product formula of iterated integrals [6, (1.5.1)]:

\[
\int_0^1 \omega_1 \cdots \omega_r \int_0^1 \omega_{r+1} \cdots \omega_{r+s} = \sum_{\sigma} \int_0^1 \omega_{\sigma(1)} \cdots \omega_{\sigma(r+s)},
\]

where \( \sigma \) ranges over all shuffles of type \( (r, s) \), that is, permutations \( \sigma \) of \( r + s \) letters with \( \sigma^{-1}(1) < \cdots < \sigma^{-1}(r) \) and \( \sigma^{-1}(r + 1) < \cdots < \sigma^{-1}(r + s) \). \( \square \)
3. Double zeta space at level $N$

Now we introduce the level $N$ version of the formal double zeta space studied in [7] as follows. Let $k \geq 2$ and $\mathcal{DZ}(N)_k$ be the $\mathbb{Q}$-vector space spanned by the formal symbols $Z^{a,b}_{r,s} = Z(N)_{r,s}^{a,b}$, $P_{r,s}^{a,b} = P(N)_{r,s}^{a,b}$ and $Z_k^a = Z(N)_{k}^a$ for $r, s \geq 1, r + s = k, a, b \in \mathbb{Z}/N\mathbb{Z}$ with the set of relations

$$P_{r,s}^{a,b} = Z_{r,s}^{a,b} + Z_{s,r}^{b,a} + \delta_{a,b}Z_{a} + \delta_{a,b}Z_{s} = \sum_{i+j=k, i,j \geq 1} \left( \binom{i-1}{r-1} Z_{i,j}^{a+b} + \binom{i-1}{s-1} Z_{i,j}^{a+b,a} \right)$$

for $r, s \geq 1, r + s = k$. Namely,

$$\mathcal{DZ}(N)_k = \frac{\mathbb{Q}(Z(N)_{r,s}^{a,b}, P(N)_{r,s}^{a,b}, Z_{k}^a : a, b \in \mathbb{Z}/N\mathbb{Z}, r, s \geq 1, r + s = k)}{\mathbb{Q}(\text{relations (3.1)})}.$$ 

Clearly

$$\mathcal{DZ}(N)_k = \frac{\mathbb{Q}(Z(N)_{r,s}^{a,b}, Z_{k}^a : a, b \in \mathbb{Z}/N\mathbb{Z}, r, s \geq 1, r + s = k)}{\mathbb{Q}(\text{relations (3.2)})},$$ 

where the defining relations are (dropping the dependence on $N$)

$$Z_{r,s}^{a,b} + Z_{s,r}^{b,a} + \delta_{a,b}Z_{k}^a = \sum_{i+j=k, i,j \geq 1} \left( \binom{i-1}{r-1} Z_{i,j}^{a+b} + \binom{i-1}{s-1} Z_{i,j}^{a+b,a} \right).$$

Recall that we often choose $0 \leq a < N$ to represent the residue class congruent to $a$ modulo $N$. Observe that when residue class $\overline{0}$ appears in any conditions involving gcd, we should use $\mathbb{N}$ to represent it. For example, $\gcd(0, 0) = \gcd(0, N) = \gcd(N, N) = N$. Now we may define $\mathcal{PDZ}(N)_k$, the formal double zeta space of pure level $N$, by restricting $Z_{k}^a$ to

$$\Omega(N) = \{ (a, b) : 0 \leq a, b < N, \gcd(a, b, N) = 1 \}$$

and $Z_{k}^a$ to $\{ a : 1 \leq a < N, \gcd(a, N) = 1 \}$ in the above. This is well defined since if $\gcd(a, b, N) = 1$, then $\gcd(a + b, a, N) = \gcd(a + b, b, N) = 1$.

Since both sides of (3.2) are invariant under $(a, b; r, s) \leftrightarrow (b, a; s, r)$, we may just take $r \leq s$. Thus, for even $k$ the group $\mathcal{DZ}(N)_k$ has $(k-1)N^2 + N$ generators and $kN^2/2$ relations. Hence

$$\dim \mathcal{DZ}(N)_k \geq \frac{(k-2)N^2 + 2N}{2}.$$ 

Similarly, for even $k$ the group $\mathcal{PDZ}(N)_k$ has $(k-1)|\Omega(N)| + \varphi(N)$ generators and $(k-1)(|\Omega(N)| + \varphi(N))/2$ relations. Hence

$$\dim \mathcal{PDZ}(N)_k \geq \frac{(k-1)(|\Omega(N)| - \varphi(N))}{2} + \varphi(N).$$

**Remark 3.1.** (a) Note that the double zeta space $\mathcal{DZ}(2)_k$ in [9] is our $\mathcal{PDZ}(2)_k$. (b) The bound in (3.3) is not sharp. For example, when $(N, k) = (3, 4)$, we have 24 generators and only 13 independent relations instead of 15. So $\dim \mathcal{PDZ}(3)_4 = 11 > 24 - 15$.

Note that the relations (3.1) (as well as (3.2)) correspond to those in Proposition 2.1 when $r, s \geq 2$; under the correspondences

$$Z(N)_{r,s}^{a,b} \longleftrightarrow \zeta_{N}^{a,b}(r,s), \quad Z(N)_{k}^{a} \longleftrightarrow \zeta_{N}^{a(k)}, \quad P(N)_{r,s}^{a,b} \longleftrightarrow \zeta_{N}^{a(r)} \zeta_{N}^{b(s)},$$

the binomial coefficients for $i = 1$ on the right vanish in both (3.1) and (3.2). For our later applications it is convenient to allow ‘divergent’ $Z(N)_{1,k-1}^{a,b}$ and $P(N)_{1,k-1}^{a,b}$, etc., and in fact the double shuffle relations in Proposition 2.1 can be extended for $r = 1$ or $s = 1$ by using a suitable regularization procedure for $Li_{1,1}^{r,s}(1, \eta)$ etc. developed in [2], which was motivated by [8].
For a comprehensive treatment of general multiple zeta values of level $N$, see our paper [12]. Specifically, in our current situation we can define the following renormalized values. Let $T$ be a formal variable.

(1) Note that $Li_1^*(1) = \zeta_*(1) = T$ and $Li_{1,1}^*(1) = \zeta_{1,1}^*(1) = T$. By (2.2) and (2.3), for $a \in \mathbb{Z}/N\mathbb{Z},$

$$\zeta_{N,*}^a(1) = \zeta_{N,1}^a(1) = \frac{1}{N} \left( T + \sum_{n=1}^{N-1} \eta^{-an} Li_1(\eta^n) \right). \quad (3.4)$$

(2) By (2.5), for $s \geq 2$ and $a, b \in \mathbb{Z}/N\mathbb{Z},$

$$\zeta_{N,*}^{a,a}(1, s) = \frac{1}{N} \left( T + \sum_{n=1}^{N-1} \eta^{-an} Li_1(\eta^n) \right) \zeta_{N}^a(s) - \zeta_{N}^a(s, 1) - \zeta_{N}^a(s + 1),$$

$$\zeta_{N,*}^{a,b}(1, s) = \frac{1}{N} \left( T + \sum_{n=1}^{N-1} \eta^{-an} Li_1(\eta^n) \right) \zeta_{N}^b(s) - \zeta_{N}^b(s, 1) \quad \text{if} \, a \neq b.$$

(3) By (2.2), for $a, b \in \mathbb{Z}/N\mathbb{Z},$

$$\zeta_{N,*}^{a,b}(1, 1) = \frac{1}{N^2} \left( T^2 - \sum_{\beta=1}^{N-1} \eta^{-b\beta} (TLi_1(\eta^{\beta}) - Li_{1,1}(\eta^{\beta}, \eta^{-\beta})) \right) \quad (3.5)$$

$$+ \sum_{\alpha=1}^{N-1} \sum_{\beta=1}^{N} \eta^{-a\alpha - b\beta} Li_{1,1}(\eta^{\alpha}, \eta^{-\alpha}).$$

(4) By (2.4), for $a, b \in \mathbb{Z}/N\mathbb{Z},$

$$\zeta_{N,1}^{a,b}(1, s) = \frac{1}{N^2} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \eta^{-a\alpha - b\beta} Li_{1,1}^{*}(\eta^{\alpha}, \eta^{-\alpha}),$$

where $Li_{1,1}^{*}(1, 1) = \frac{1}{2} T^2$, $Li_1^{*}(\eta^a, \eta^{-\alpha}) = Li_{1,1}(\eta^a, \eta^{\beta})$ for all $a \neq 0 \in \mathbb{Z}/N\mathbb{Z}$, and

$$Li_{1,1}(1, \eta^b) = T Li_1(\eta^b) - \sum_{i=2}^{s} Li_{1,1-i,1}(1, \eta^b) - Li_{i,1}(\eta^b, \eta^{-\beta}) \quad \forall (s, \beta) \neq (1, 0).$$

The equations in Proposition 2.1 are valid for all $r, s \geq 1$. Here we have used the fact that, for $\alpha, \beta \in \mathbb{Z}/N\mathbb{Z}$ ($\alpha \neq 0$), we have

$$Li_1(\eta^a) = \int_0^1 \frac{\eta^a t}{1 - \eta^a t} dt, \quad Li_{1,1}(\eta^a, \eta^{-\alpha}) = \int_{1 \leq t_1 > t_2 > 0} \frac{\eta^a t_1}{1 - \eta^a t_1} \frac{\eta^2 dt_2}{1 - \eta^2 t_2}. \quad (3.5)$$

With the above regularized values we can now extend Proposition 2.1 to these cases. For $r \geq 2$ and $s \geq 1$, we set $\zeta_{N,*}^a(r) = \zeta_{N,1}^a(r) = \zeta_{N}^a(r)$ and $\zeta_{N,*}^{a,b}(r, s) = \zeta_{N,1}^{a,b}(r, s) = \zeta_{N}^{a,b}(r, s)$.

**PROPOSITION 3.2.** For positive integers $r, s \geq 1$ and $a, b \in \mathbb{Z}/N\mathbb{Z}$, we have

$$\zeta_{N,*}^{a,b}(r, s) = \zeta_{N,*}^{a,b}(r, s) + \zeta_{N,*}^{b,a}(s, r) + \delta_{a,b} \zeta_{N,*}^{a}(s + r),$$

$$\zeta_{N,1}^{a,b}(r, s) = \sum_{i+j=r+s \atop i, j \geq 0} \left( \binom{i-1}{r-1} \zeta_{N,1}^{a+b}(i, j) + \binom{i-1}{s-1} \zeta_{N,1}^{a+b}(i, j) \right),$$

where we set $\binom{0}{0} = 1$. 

Proof. We need to check the relations only in the cases when \( r = 1, s \geq 2 \) or \( r \geq 2, s = 1 \) or \( r = s = 1 \). These follow directly from the definitions and the stuffle and shuffle relations among \( Li_1 \) and \( Li_{1,1} \). We leave the details to the interested reader.

The following theorem generalizes both a result of \[9\] and a result of \[7\].

**Theorem 3.3.** Let \( k \) be a positive even integer and \( a \in \mathbb{Z}/N\mathbb{Z} \). Then

\[
\begin{align*}
\sum_{1 \leq r < k, \, r \text{ odd}} Z_{r,k}^{a,a} &= \frac{1}{4}(2Z_{1,k-1}^{a,N} + 2Z_{1,k-1}^{2a,a} - Z_k^a + 2\delta_{a,0}Z_k^N), \\
\sum_{1 < r < k, \, r \text{ even}} Z_{r,k}^{a,a} &= \frac{1}{4}(2Z_{1,k-1}^{a,N} - 2Z_{1,k-1}^{2a,a} + Z_k^a + 2\delta_{a,0}Z_k^N).
\end{align*}
\]

**Proof.** Consider the generating functions

\[
Z_k^{a,b}(X,Y) = \sum_{r+s=k} Z_{r,s}^{a,b}X^{r-1}Y^{s-1}.
\]

By (3.2), we see that

\[
Z_k^{a,b}(X,Y) + Z_k^{b,a}(Y,X) + \delta_{a,b}Z_k^a \frac{X^{k-1} - Y^{k-1}}{X - Y} = Z_k^{a+b,b}(X + Y, Y) + Z_k^{a+b,a}(X + Y, X).
\]

Setting \((X,Y) = (1,0)\) and then \((X,Y) = (1,-1)\), we get, respectively,

\[
\begin{align*}
Z_{k-1,1}^{a,b} + Z_{1,k-1}^{b,a} + \delta_{a,b}Z_k^a &= Z_k^{a+b,b} + \sum_{r=1}^{k-1} Z_{r,k-r}^{a+b,a}, \\
\sum_{r=1}^{k-1} (-1)^{r-1}(Z_{r,k-r}^{a,b} + Z_{r,k-r}^{b,a}) + \delta_{a,b}Z_k^a &= Z_{1,k-1}^{a+b,b} + Z_{1,k-1}^{a+b,a}.
\end{align*}
\]

Setting \( b = N \) in (3.8) and \( a = b \) in (3.9), we get

\[
\begin{align*}
\sum_{r=1}^{k-1} Z_{r,k-r}^{a,a} &= Z_{1,k-1}^{a,N} + \delta_{a,0}Z_k^N, \\
2\sum_{r=1}^{k-1} (-1)^{r-1}Z_{r,k-r}^{a,a} + Z_k^a &= 2Z_{1,k-1}^{2a,a}.
\end{align*}
\]

By adding (respectively, subtracting) twice (3.10) to (respectively, from) (3.11), we obtain (3.6) and (3.7).

**Remark 3.4.** Part (1) of [9, Theorem 1] follows from the special case of \( N = 2 \) and \( a = 1 \) of our theorem. By taking \( N = a = 1 \) in the theorem, we obtain [7, Theorem 1].

Next we describe the linear relations among the formal variables \( Z_{i,j}^{a,b} \) using some homogeneous polynomials.

**Proposition 3.5.** Let \( k \geq 2 \) be a positive integer. Let \( e_{i,j}^{a,b} \in \mathbb{Q} \) for all \( i, j \in \mathbb{N} \) and \( a, b \in \mathbb{Z}/N\mathbb{Z} \). Then the following two statements are equivalent.

(i) The relation

\[
\sum_{0 \leq a < b < N} \sum_{i+j=k} e_{i,j}^{a,b} Z_{i,j}^{a,b} \equiv 0 \pmod{\mathbb{Q}(Z_k^a : a \in \mathbb{Z}/N\mathbb{Z})}
\]

holds in \( \mathcal{DZ}(N)_k \). Here and in the rest of the paper, \( \sum_{i+j=k} \) means \( \sum_{i=j=k, i,j \geq 1} \).
(i) When $\cd$ determines the values should come from a linear combination of (3.2) with various choices of $\deg \cd$.

Then we have the following lemma.

Further, the following two statements are equivalent.

(iii) The relation

$$
\sum_{a,b \in \Omega(N), a \leq b} c_{i,j}^{a,b} Z_{i,j}^{a,b} \equiv 0 \pmod{\mathbb{Q}(Z_k : 1 \leq a < N, \gcd(a, N) = 1)}
$$

holds in $\mathcal{PDZ}(Nk)$.

(iv) There exist some homogeneous polynomials $F_{a,b} \in \mathbb{Q}[X,Y]$ $(a, b \in \Omega(N)$ and $a \leq b)$ of degree $k - 2$ such that

$$
\sum_{a,b \in \Omega(N)} F_{a,b}(X_b, Y_a) + F_{a,b}(Y_a, X_b) - F_{a,b}(X_{a+b} + Y_b, X_{a+b})
- F_{a,b}(X_{a+b} + X_{a+b} + Y_a) = \sum_{a,b \in \Omega(N), i+j=k} \left(k-2\right) c_{i,j}^{a,b} X_i^{a-1} Y_j^{b-1}. 
$$

Proof. For any fixed $a, b \in \mathbb{Z}/N\mathbb{Z}$ we take $F_{a,b}(X, Y) = \left(k-2\right) X^{a-1} Y^{b-1}$ $(r+s = k)$ and $F_{c,d}(X, Y) = 0$ for all $(c, d) \not= (a, b)$. Then the expansion of the left-hand side of (3.13) determines the values $c_{i,j}^{a,b}$ uniquely such that (3.2) holds, which implies (i). In fact, when $a \not= b$, we obtain an exact equation in (i). Since any relation of the form in (i) in $\mathcal{DZ}(N_k)$ should come from a linear combination of (3.2) with various choices of $(a, b) \in (\mathbb{Z}/N\mathbb{Z})^2$ modulo $\mathbb{Q}(Z_k : a \in \mathbb{Z}/N\mathbb{Z})$ and any homogeneous polynomial is a linear combination of monomials of the form $F_{a,b}(X, Y)$, the equivalence of (i) and (ii) follows immediately. Similar arguments clearly show the equivalence of (iii) and (iv) \qed

Remark 3.6. Proposition 3.5 generalizes [7, Proposition 2.2(i)(ii): 9, Lemma 1]. In fact, when $N = 2$, we can take $(a, b) = (0, 1), (1, 1)$ in (iii) and (iv) of Proposition 3.5; then we see that $F = F_{0,1}$ and $G = F_{1,1}$ in [7, Proposition 2.2(ii)] with relabeling of the variables as follows: $X_0 Y_1^j \rightarrow X_1 Y_1^j$, $X_1 Y_0^j \rightarrow X_2 Y_2^j$ and $X_1 Y_1^j \rightarrow X_3 Y_3^j$.

4. Fourier series expansion of the double Eisenstein series at level $N$

In this section, we will describe a procedure to find the Fourier series expansion of the double Eisenstein series. This can be generalized to larger depths. Using similar notation to that in [3, 7], for any $a \in \mathbb{Z}/N\mathbb{Z}$ and positive integer $s$ set

$$
\Psi_s^a(\tau) = \Psi_s^{a,N}(\tau) = \sum_{c \equiv a \pmod{N}, c \in \mathbb{Z}} \frac{1}{(\tau + c)^s} \quad \forall s \geq 2,
$$

$$
\Psi_1^a(\tau) = \Psi_1^{a,N}(\tau) = \lim_{M \to \infty} \sum_{c \equiv a \pmod{N}, |c| < M} \frac{1}{\tau + c}.
$$

Then we have the following lemma.
Lemma 4.1. Let \( q = e^{2\pi i\tau} \) and \( \eta = \eta_N = \exp(2\pi i/N) \). Then, for any \( a \in \mathbb{Z}/N\mathbb{Z} \) and \( s \in \mathbb{N} \), we have

\[
\Psi_a^s(N\tau) = \begin{cases} 
-\frac{\pi i}{N} - \frac{2\pi i}{N} \sum_{n \geq 1} \eta^a q^n & \text{if } s = 1; \\
\frac{(-2\pi i)^s}{N^s(s-1)!} \sum_{n \geq 1} n^{s-1} \eta^a q^n & \text{if } s \geq 2.
\end{cases}
\]  

(4.1)

Proof. The well-known Lipschitz formula implies, for all \( k \geq 2 \),

\[
\sum_{n \in \mathbb{Z}} \frac{1}{(x+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} e^{2\pi i n x}.
\]

Thus, by setting \( x = \tau + a/N \), we get

\[
\Psi_a^k(N\tau) = \sum_{n \in \mathbb{Z}} \frac{1}{(N\tau + nN + a)^k} = \frac{(-2\pi i)^k}{N^k(k-1)!} \sum_{n \geq 1} n^{k-1} \eta^a e^{2\pi i n x},
\]

as desired.

Now we deal with the special case when \( s = 1 \). In the Summation Theorem [10, p. 305], we take \( f(z) = 1/(z + \tau') \), where \( \tau' = (\tau + a)/N \). Then we get

\[
\lim_{M \to \infty} \sum_{c \equiv a (\text{mod } N), |c| < M} \frac{1}{\tau + c} = \frac{1}{N} \lim_{M \to \infty} \sum_{n = -M}^{M} \frac{1}{\tau' + n} = -\text{Res}_{z=-\tau'} \left( \frac{\pi \cot \pi z}{z + \tau'} \right).
\]

Since

\[
\cot \pi z = (-i) \frac{1 + e^{2\pi iz}}{1 - e^{2\pi iz}} = -i - 2i \sum_{n \geq 1} e^{2\pi i n z},
\]

we have

\[
\Psi_1^0(N\tau) = \frac{\pi}{N} \cot \left( \pi \left( \tau + \frac{a}{N} \right) \right) = -\frac{\pi i}{N} - \frac{2\pi i}{N} \sum_{n \geq 1} \eta^a q^n.
\]

This completes the proof of the lemma. \( \square \)

Corollary 4.2. For any \( a = (a_1, \ldots, a_d) \in (\mathbb{Z}/N\mathbb{Z})^d \) and \( s = (s_1, \ldots, s_d) \in \mathbb{N}^d \) set

\[
g^a_s(\tau) = g^a_s(N\tau) = \sum_{m_1 > \cdots > m_d > 0} \prod_{j=1}^{d} \Psi_{s_j}^{a_j}(m_j N\tau).
\]  

(4.2)

Then we have

\[
g^a_s(\tau) = \frac{(-2\pi i)^{|s|}}{N^{|s|}(s-1)!} \sum_{n=1}^{\infty} \sigma^a_{s-1}(n) q^n,
\]

(4.3)

where \( |s| = s_1 + \cdots + s_d \), \( s-1 = \prod_{j=1}^{d} (s_j - 1)! \) and \( s - 1 = (s_1 - 1, \ldots, s_d - 1) \).
It is now easy to decompose the level-$N$ Eisenstein series (of one variable) into the following form:

\[ G_a^r(\tau) = \zeta_a^r(N) + g_a^r(\tau) \quad (4.4) \]

for any positive integer \( r \geq 3 \) and \( a \in \mathbb{Z}/N\mathbb{Z} \). So we can define two extensions of \( G_a^r(\tau) \) as follows.

**Definition 4.3.** Let \( \sharp = \ast \) or \( * \). For all \( s \geq 1 \), we define

\[ G_{a \sharp}^r(\tau) = \zeta_a^r(s) + g_{a \sharp}^r(\tau) \quad (4.5) \]

Note that the definition is independent of whether \( \sharp = \ast \) or \( * \) by (3.4).

The following theorem is the key to the double shuffle relations satisfied by the double Eisenstein series at level \( N \).

**Theorem 4.4.** The Fourier series expansion of \( G_{r,s}^{a,b}(\tau) \) for \( r \geq 3, \ s \geq 2 \) is given by

\[ G_{r,s}^{a,b}(\tau) = \zeta_{N}^{a,b}(r,s) + g_{r,s}^{a,b}(\tau) + \sum_{h \geq 1, \ p \geq \min\{r,s\}} \zeta_{N}^{a-b}(p) \left(-1\right)^s \left( \frac{p-1}{s-1} \right) g_h^b(\tau) + \left(-1\right)^p \left( \frac{p-1}{r-1} \right) g_h^a(\tau). \quad (4.6) \]

**Proof.** Our proof follows the lines of that of [7, Theorem 6]. We decompose \( G_{r,s}^{a,b}(\tau) \) into the sum of the following four types: (1) \( m = n = 0 \), (2) \( m > n = 0 \), (3) \( m = n > 0 \) and (4) \( m > n > 0 \).

**Case 1:** \( m = n = 0 \). It gives rise to exactly

\[ \sum_{c,d > 0 \atop c \equiv a, d \equiv b (\mod N)} \frac{1}{c^s d^r} = \zeta_{N}^{a,b}(r,s). \]

**Case 2:** \( m > n = 0 \). We are looking at

\[ \sum_{m > 0, \ d > 0 \atop c \equiv a, d \equiv b (\mod N)} \frac{1}{(mN\tau + c)^r d^s} = \sum_{m > 0} \Psi_{r}^{a}(mN\tau) \sum_{d > 0 \atop d \equiv b (\mod N)} \frac{1}{d^s} = g_{r}^{a}(\tau)\zeta_{N}^{b}(s). \]

**Case 3:** \( m = n > 0 \). Then we write

\[ \sum_{c \equiv a, \ d \equiv b (\mod N)} \frac{1}{(mN\tau + c)^r (mN\tau + d)^s} = \sum_{m > 0} \Psi_{r,s}^{a,b}(mN\tau). \]

Next we compute \( \Psi_{r,s}^{a,b}(\tau) \). Using the partial fraction

\[ \frac{1}{(\tau + c)^r(\tau + d)^s} = \sum_{h \geq 1, \ p \geq \min\{r,s\}} \left( \frac{-1}{r} \left( \frac{p-1}{s-1} \right) \frac{1}{c-d} \right)^h + \sum_{h \geq 1, \ p \geq \min\{r,s\}} \left( \frac{-1}{s} \left( \frac{p-1}{r-1} \right) \frac{1}{c-d} \right)^h \]

,
we obtain
\[ \Psi_{a,b}^s(\tau) = \sum_{c \equiv a, \; d \equiv b \pmod{N}} \frac{1}{(\tau + c)^r(\tau + d)^s} \]
\[ = \sum_{h + p = r + s, \; c \equiv a, \; d \equiv b \pmod{N}} \left[ (-1)^s \left( \frac{p - 1}{s - 1} \right) \frac{1}{(c - d)^p(\tau + c)^s} \right. \\
+ (-1)^{p-r} \left( \frac{p - 1}{r - 1} \right) \frac{1}{(c - d)^p(\tau + d)^s} \left] \right. \\
= \sum_{h + p = r + s, \; h \geq 1, \; p \geq \min\{r, s\}} \left[ (-1)^s \left( \frac{p - 1}{s - 1} \right) \Psi_{h}^a(\tau) + (-1)^{p-r} \left( \frac{p - 1}{r - 1} \right) \Psi_{h}^b(\tau) \right]. \]

Hence
\[ \sum_{m > 0} \frac{1}{(mN\tau + c)^r(mN\tau + d)^s} = \sum_{m > 0} \Psi_{r,s}^a(mN\tau) \]
\[ = \sum_{h + p = r + s, \; h \geq 1, \; p \geq \min\{r, s\}} \left[ (-1)^s \left( \frac{p - 1}{s - 1} \right) g_{h}^a(\tau) + (-1)^{p-r} \left( \frac{p - 1}{r - 1} \right) g_{h}^b(\tau) \right]. \]

Here the special case \( h = 1 \) has to be treated carefully by using (4.1).

Case 4: \( m > n > 0 \). We have
\[ \sum_{m > n > 0} \frac{1}{(mN\tau + c)^r(nN\tau + d)^s} = g_{r,s}^a(\tau). \]

The theorem follows by summing the above four parts.

\[ \square \]

Motivated by (4.6), we now have the extension of the double Eisenstein series of level \( N \) to the following regularized form.

**Definition 4.5.** Let \( \sharp = \overset{\cdot}{\overset{\cdot}{\cdot}} \) or \( * \). Then, for all \( r, s \geq 1 \), we define
\[ G_{r,s}^{a,b}(\tau) = \zeta_{N,\sharp}^{a,b}(r, s) + g_{N,\sharp}^a(\tau) + g_{N,\sharp}^b(\tau) \]
\[ + \sum_{h + p = r + s, \; h \geq 1, \; p \geq \min\{r, s\}} \zeta_{N,\sharp}^{a-b}(p) \left[ (-1)^s \left( \frac{p - 1}{s - 1} \right) g_{h}^a(\tau) + (-1)^{p-r} \left( \frac{p - 1}{r - 1} \right) g_{h}^b(\tau) \right]. \] (4.7)

**Remark 4.6.** Unlike Definition 4.3 this definition of \( G_{r,s,\sharp}^{a,b}(\tau) \) depends on the choice of the regularization scheme \( \sharp \). Moreover, because of (3.4) the dependence appears only in the constant term \( \zeta_{N,\sharp}^{a,b}(r, s) \).

If \( s, r \geq 3 \) and \( a, b \in \mathbb{Z}/N\mathbb{Z} \), then it is not hard to show
\[ G_{r}^{a}(\tau)G_{s}^{b}(\tau) = G_{r,s}^{a,b}(\tau) + G_{r,s}^{b,a}(\tau) + \delta_{a,b}G_{r+s}^{a}(\tau) \] (4.8)
which follows easily by the definition. But
\[ G_{r}^{a}(\tau)G_{s}^{b}(\tau) \neq \sum_{i+j=r+s, \; i,j \geq 1} \binom{i-1}{p-1} \binom{j-1}{s-1} G_{i,j}^{a+b,b}(\tau) + \binom{i-1}{p-1} G_{i,j}^{a+b,a}(\tau) \]
since the right-hand side has undefined terms. Our goal is to give an extension of these double shuffle relations to the case \( r, g \geq 1 \) and \( (r, s) \neq (1, 1) \) by using a complete version of the zeta values and the Eisenstein series of level \( N \).

5. Decomposition of the zeta values at level \( N \)

In this section, we break \( \zeta^a_N(s) \) into two parts, one of which is inspired by its complete version defined as follows. For all positive integers \( n \) and \( a \in \mathbb{Z}/N\mathbb{Z} \), we set

\[
\zeta^a_N(n) = \frac{1}{2} \sum_{\substack{k \in \mathbb{Z} \neq 0 \atop k \equiv a \pmod{N}}} \frac{1}{k^n} = \frac{1}{2} \lim_{M \to \infty} \sum_{0 < |k| < M \atop k \equiv a \pmod{N}} \frac{1}{k^n}
\]

by using the Cauchy principal value. This infinite series converges absolutely for \( n \geq 2 \) and conditionally for \( n = 1 \). This complete version of \( \zeta^a_N(s) \) clearly satisfies the shuffle relations as given by (2.5).

**Remark 5.1.** When the level \( N = 2 \), the decomposition of \( \zeta^a_N(n) \) corresponds to the decomposition of it into the Bernoulli number part and non-Bernoulli number part. See [9, p. 1103]. If \( a \equiv 0 \pmod{2} \), then the non-Bernoulli number part is essentially the Riemann zeta values at odd integers.

To extract more information from \( \zeta^a_N(n) \) and find its relation to \( \zeta^a_N(n) \), we now recall that the \( n \)th Bernoulli periodic function \( \bar{B}_n(x) \) (see [1, p. 267]) has the series expansion

\[
\bar{B}_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k \in \mathbb{Z} \neq 0} \frac{e^{2\pi ikx}}{k^n} = -\frac{n!}{(2\pi i)^n} \lim_{M \to \infty} \sum_{0 < |k| < M} \frac{e^{2\pi ikx}}{k^n},
\]

which converges absolutely for \( n \geq 2 \) and conditionally for \( n = 1 \). It is related to the Bernoulli polynomials by \( \bar{B}_n(x) = B_n(\{x\}) \), where \( \{x\} \) is the fractional part of \( x \), except for \( n = x = 1 \) when \( B_1(\{1\}) = B_1(0) = -1/2 \) while

\[
\bar{B}_1(1) = -\frac{1}{2\pi i} \lim_{M \to \infty} \sum_{0 < |k| < M} \frac{1}{k} = 0
\]

by symmetry. The following identity motivates our definition of the constant term of the generating series for the level-\( N \) Eisenstein series.

**Proposition 5.2.** For all \( n \geq 1 \) and \( a \in \mathbb{Z}/N\mathbb{Z} \), we have

\[
\zeta^a_N(n) = -\frac{(2\pi i)^n}{2N \cdot n!} \sum_{l=1}^N \exp \left(-\frac{2\pi ila}{N}\right) \bar{B}_n \left( \frac{l}{N} \right).
\]

**Proof.** This follows quickly from the identity

\[
\sum_{l=1}^N \eta^l_N = \begin{cases} N & \text{if } N|m, \\ 0 & \text{otherwise,} \end{cases}
\]

for any integer \( m \).

**Corollary 5.3.** Let \( 1 \leq a \leq N \). Then, for all \( r \geq 1 \),

\[
\sum_{l=1}^N \sin \left( \frac{2\pi la}{N} \right) \bar{B}_{2r} \left( \frac{l}{N} \right) = \sum_{l=1}^N \cos \left( \frac{2\pi la}{N} \right) \bar{B}_{2r+1} \left( \frac{l}{N} \right) = 0.
\]
Proof. Note that under $a \rightarrow N - a$ the left-hand side of (5.4) is invariant if $n$ is even and changes the sign if $n$ is odd. It also follows from the fact that $B_{2r}(1 - x) = B_{2r}(x)$ and $B_{2r+1}(1 - x) = -B_{2r+1}(x)$ for all $r \geq 0$.

The following corollary provides the exact relation between $\zeta_N^a$ and $\zeta_N^a$.

**Corollary 5.4.** For all positive integers $n$ and $a \in \mathbb{Z}/N\mathbb{Z}$, we have

$$g_N^a(n) = \frac{1}{2}(\zeta_N^{a;2}(n) + (-1)^n \zeta_N^{-a;2}(n)),$$

where $\sharp = m$ or $\star$, and when $n = 1$ the right-hand side is defined by (3.4).

Proof. Suppose $n \geq 2$ first. Then we can break the sum in (5.1) into two parts, one with positive indices, which produces $(2\pi i)^{-n}\zeta_N^a(n)$ and the other negative, which leads to $(-2\pi i)^{-n}\zeta_N^{-a}(n)$. For $n = 1$, the corollary follows easily from Proposition 5.2 by using the fact that

$$Li_1(-e^{2\pi i \theta}) - Li_1(e^{2\pi i \theta}) = 2\pi i \theta - \pi i = 2\pi i B_1(\theta) \quad \forall \theta \in (0, 1).$$

Note that the $l = N$ term in the sum on the right-hand side of (5.4) vanishes by (5.3). We leave the details to the interested reader.

Proposition 5.2 leads us to the following definition if we follow the guideline that the constant term of the multiple Eisenstein series (even for the regularized values) should be closely related to the multiple zeta values, at any level.

**Definition 5.5.** For $n > 0$ and $1 \leq a \leq N$, we define

$$\beta_n^a = \beta_n^{a;N} = -\frac{1}{2N \cdot n!} \sum_{l=1}^{N} \exp(-\frac{2\pi ila}{N}) B_n\left(\frac{l}{N}\right).$$

(5.5)

It is clear from Corollary 5.4, Proposition 5.2 and (5.3) that

$$\beta_n^a = (2\pi i)^{-n} \beta_N^a(n) + \delta_{n,1} \frac{\zeta_N^a}{4N}. \quad (5.6)$$

Let $\sharp = m$ or $\star$. For all $r \geq 1$, $s \geq 1$ and $a, b \in \mathbb{Z}/N\mathbb{Z}$, we define

$$g_r^a(q) = (2\pi i)^{-r} g_N^a(q), \quad \beta_r^a = (2\pi i)^{-r} \left(\zeta_N^{a;2}(r) - (-1)^r \zeta_N^{-a;2}(r)\right) - \delta_{r,1} \frac{\zeta_N^a}{4N}. \quad (5.7)$$

$$I_{r,s}^{a,b}(q) = g_r^a(q) \beta_s^b + \sum_{h+p=r+s \geq 1} \tilde{\beta}_{p}^{a-b} \left((-1)^{s} \frac{p-1}{s-1} \tilde{g}_h^a(q) + (-1)^{p-r} \frac{p-1}{r-1} \tilde{g}_h^b(q)\right).$$

Note that the quantities defined above are independent of whether $\sharp = m$ or $\star$ according to (3.4). The following result generalizes [9, Lemma 3].

**Proposition 5.6.** For any integer $k \geq 2$ and $a, b \in \mathbb{Z}/N\mathbb{Z}$, define the generating series

$$\gamma_k^{a,b}(X, Y) = \sum_{r+s=k} I_{r,s}^{a,b}(q) X^{r-1} Y^{s-1}.$$ 

If $k \geq 3$, then

$$\sum_{h+p=k \geq 1} \frac{X^{h-1} Y^{p-1} \tilde{g}_h^a(q) \beta_p^b + Y^{h-1} X^{p-1} \tilde{g}_h^b(q) \beta_p^a}{h+p \geq k} = \gamma_k^{a,b}(X, Y) + \gamma_k^{b,a}(Y, X) = \gamma_k^{a+b,a}(X + Y, X) + \gamma_k^{a+b,b}(X + Y, Y). \quad (5.7)$$
If \( k = 2 \) (that is, \( r = s = 1 \)), then we have
\[
\tilde{g}_1^a(q)\beta_1^b + \tilde{g}_1^b(q)\beta_1^a = I_{1,1}^{a+b,b} + I_{1,1}^{a+b,a}.
\] (5.8)

Proof. We first see that
\[
\mathcal{I}_{k}^{a,b}(X,Y) = \sum_{r+s=k} I_{r,s}^{a,b}(q)X^{r-1}Y^{s-1}
\]
\[
= \sum_{h+p=k \atop h \geq 1} \tilde{g}_h^a(q)\beta_p^b X^{h-1}Y^{p-1} + \sum_{h+p=k} \tilde{\beta}_p^{a-b}(p)
\]
\[
\cdot \left[ \tilde{g}_h^a(q) \left( \sum_{r+s=k} (-1)^s \binom{p-1}{s-1} X^{r-1}Y^{s-1} \right) \right.
\]
\[
+ \tilde{g}_h^b(q) \left( \sum_{r+s=k} (-1)^{p-r} \binom{p-1}{r-1} X^{r-1}Y^{s-1} \right) \right]
\]
\[
= \sum_{h+p=k \atop h \geq 1} \tilde{g}_h^a(q)\beta_p^b X^{h-1}Y^{p-1}
\]
\[
+ \sum_{h+p=k} \tilde{\beta}_p^{a-b}(q^{h-1}(X-Y)^{p-1} - \tilde{g}_h^a(q)X^{h-1}(X-Y)^{p-1})
\] (5.9)

from the binomial expansion. Now by considering even and odd \( p \), we see that the sum term of (5.9) has no contribution in \( \mathcal{I}_{k}^{a,b}(X,Y) + \tilde{g}_h^a(q)^{2}\). The last equality of (5.7) is straightforward, so we omit its proof. Finally, (5.8) follows easily by direct computation. \( \square \)

6. Double shuffle relations of the double Eisenstein series at level \( N \)

In this section, we are going to define three power series \( E_{r,s}^{a,b}(q), P_{r,s}^{a,b}(q) \) and \( E_{k}(q) \) which, together with \( I_{r,s}^{a,b}(q) \) for the first one, are complementary to the double zeta values, product of the zeta values and the zeta values at level \( N \), respectively. The latter values, essentially the constant terms of the corresponding Eisenstein series, satisfy the double shuffle relations in Proposition 2.1.

Write \( q = \exp(2\pi i\tau) \) and define
\[
\tilde{g}_k^a(q) = \tilde{g}_k^{a:N}(q) = (2\pi i)^{-|s|}\tilde{g}_k^{a:N}(N\tau), \quad \tilde{g}_k^a(q) = \tilde{g}_k^{a:N}(q) = (2\pi i)^{-|s|}g_k^a(\tau),
\]
\[
(\tilde{g}_k^a)'(q) = (\tilde{g}_k^{a:N})'(q) = -\sum_{n=1}^{\infty} n\tilde{g}_k^a(q^n), \quad \left( f'_k(q) = \frac{q}{Nk dq} f_k(q) \right),
\]
\[
\beta_{r,s}^{a,b}(q) = \tilde{g}_r^a(q)\beta_s^b + \sum_{i+j=r+s} \beta_i^{a-b} \left[ (-1)^s \binom{i-1}{s-1} \tilde{g}_j^a(q) + (-1)^{i-r} \binom{i-1}{r-1} \tilde{g}_j^b(q) \right],
\]
\[
\varepsilon_{r,s}^{a,b}(q) = \delta_{r,2}(\tilde{g}_r^a)'(q) - \delta_{r,1}(\tilde{g}_r^{a:N})'(q) + \delta_{s,1}(\tilde{g}_s^b)'(q) + \tilde{g}_s^a(q) + N\delta_{r,1}\delta_{s,1} \gamma_N^{a,b}(q),
\]
where \( \gamma_N^{a,b} = \gamma_N^{a,b}(q) \) can be defined by the procedure to be outlined in Theorem 8.1. Further, we set
\[
f_2^a = f_2^a(q) = \frac{1}{N^2} \sum_{n,u=1}^{\infty} \eta_{au} nq^{nu} = \frac{1}{N^2} \sum_{m=1}^{\infty} \kappa_1^a(m)q^m, \quad \text{where } \kappa_1^a(m) = \sum_{n,u=m} \eta_{au} n.
\] (6.1)

Remark 6.1. (i) To save space, in the rest of the paper we will always suppress the dependence on \( q \) in the \( q \)-series \( \gamma_N^{a,b}(q), f_2^a(q), \) etc. Of course, they all depend on \( N \).
(ii) We will see that the definition of \( \gamma_{N}^{a,b} \) is not unique. For example, for small levels we may define \( \gamma_{N}^{a,b} \) explicitly as follows. For \( N = 1 \), \( \gamma_{1}^{0,0} = f_{0}^{0} = \tilde{g}_{0}^{0} \). For \( N = 2 \), we can set

\[
\begin{align*}
\gamma_{2}^{0,0} &= f_{2}^{0} = \tilde{g}_{2}^{0}, & \gamma_{2}^{0,1} &= \gamma_{2}^{1,0} = 0, & \gamma_{2}^{1,1} &= \tilde{g}_{2}^{1} - f_{2}^{1}.
\end{align*}
\]

When \( N = 2 \), our choice of the above is different from that of \([9]\) (see Remark 8.4). For \( N = 3 \), we may define

\[
\begin{align*}
\gamma_{3}^{a,a} &= \tilde{g}_{2}^{a} - f_{2}^{a} & \text{and} & \gamma_{3}^{a,0} &= f_{2}^{a} + f_{2}^{a} - \gamma_{3}^{a,a} & \text{for } a = 1, 2, \\
\gamma_{3}^{0,0} &= \tilde{g}_{2}^{0}, & \gamma_{3}^{0,1} &= \gamma_{3}^{1,0} = \gamma_{3}^{2,0} = \gamma_{3}^{1,2} = \gamma_{3}^{2,1} = 0.
\end{align*}
\]

**Definition 6.2.** We define

\[
E_{k}^{a}(q) = E_{k}^{a;b,N}(q) = \begin{cases} 
\tilde{g}_{k}^{a}(q) & \text{if } k > 2, \\
0 & \text{if } k \leq 2;
\end{cases}
\]

\[
E_{r,s}^{a,b}(q) = E_{r,s}^{a,b;N}(q) = \tilde{g}_{r,s}^{a,b}(q) + \beta_{r,s}^{a,b}(q) + \frac{\varepsilon_{r,s}^{a,b}(q)}{2N}, \quad r, s \geq 1;
\]

\[
P_{r,s}^{a,b}(q) = P_{r,s}^{a,b;N}(q) = \tilde{g}_{r,s}^{a}(q)\tilde{g}_{s}^{b}(q) + \beta_{r,s}^{a}(q)\tilde{g}_{s}^{b}(q) + \beta_{s}^{b}(q)g_{r,s}^{a}(q) + \delta_{r,s}^{a}(q) + \delta_{s,1}(q)g_{r,s}^{1}(q) + \delta_{r,1}(q)g_{s,1}(q), \quad r, s \geq 1.
\]

The quantities \( \lambda_{N}^{a,b} = \lambda_{N}^{a,b}(q) = \lambda_{N}^{b,a}(q) \) will be defined by the procedure to be outlined in Theorem 8.1 together with the quantities \( \gamma_{N}^{a,b} \).

For example, to be compatible with (6.2), we set

\[
\lambda_{2}^{0,0} = \lambda_{2}^{1,0} = 0, \quad \lambda_{2}^{1,1} = -f_{2}^{1},
\]

and to be compatible with (6.3), we set

\[
\lambda_{3}^{0,0} = \lambda_{3}^{1,0} = \lambda_{3}^{2,0} = \lambda_{3}^{1,1} = -f_{2}^{1}, \quad \lambda_{3}^{2,2} = -f_{2}^{2}.
\]

Roughly speaking, the double Eisenstein series \( G_{r,s}^{a,b}(\tau) \) is given by the sum of \( \zeta_{r,s}^{a,b}(\tau, s) \) and \((2\pi i)^{r+s}E_{r,s}^{a,b}(q)\) (similar for the depth-1 Eisenstein series \( G_{r,s}^{a}(\tau) \)) while the product \( G_{r,s}^{a}(\tau)G_{s}^{b}(\tau) \) is given by the sum of \( \zeta_{r,s}^{a,b}(\tau, b) \) and \((2\pi i)^{+}\tilde{g}_{r,s}^{a,b}(q)\). By Proposition 2.1, the zeta function part already satisfies the double shuffle relations. So to prove that similar relations hold for the Eisenstein series, it suffices to prove the next result, which generalizes \([9, \text{Lemma 4}]\).

**Theorem 6.3.** Let \( N \) be a positive integer. Then there are suitable choices of \( \gamma_{N}^{a,b} \) and \( \lambda_{N}^{a,b} \), \( 0 \leq a, b < N \), such that they provide the solution to the linear system

\[
\gamma_{N}^{a,a} - \lambda_{N}^{a,a} = \tilde{g}_{2}^{a}, \quad \gamma_{N}^{a,b} + \gamma_{N}^{b,a} - 2\lambda_{N}^{a,b} = 0 \quad \forall a \neq b \in \mathbb{Z}/N\mathbb{Z},
\]

(6.6)

and

\[
\gamma_{N}^{a,b} + \gamma_{N}^{b,a} - 2\lambda_{N}^{a,b} = f_{2}^{a} + f_{2}^{b} \quad \forall a, b \in \mathbb{Z}/N\mathbb{Z}.
\]

Consequently, for all \( r, s \geq 1 \) the three power series \( E_{r,s}^{a,b}(q) \), \( P_{r,s}^{a,b}(q) \) and \( E_{k}^{a}(q) \) satisfy the double shuffle relation at level \( N \):

\[
P_{r,s}^{a,b}(q) = E_{r,s}^{a,b}(q) + E_{s,r}^{b,a}(q) + \delta_{a,b}E_{r+s}^{a,b}(q)
\]

\[
= \sum_{i+j=r+s} \left( \binom{i-1}{r-1} E_{i,j}^{a,b}(q) + \binom{i-1}{s-1} E_{i,j}^{a,b}(q) \right).
\]

(6.8)
Proof. It suffices to show that there are suitable choices of $\gamma_{N}^{a,b}$ and $\lambda_{N}^{a,b}$ satisfying (6.6) and (6.7) such that the generating functions

$$\mathbf{e}^{a}(X) = \sum_{k \geq 1} E_{k}(q)X^{k-1},$$

$$\mathbf{e}^{a,b}(X, Y) = \sum_{r,s \geq 1} E_{r,s}^{a,b}(q)X^{r-1}Y^{s-1},$$

$$\mathbf{p}^{a,b}(X, Y) = \sum_{r,s \geq 1} P_{r,s}^{a,b}(q)X^{r-1}Y^{s-1},$$

satisfy the double shuffle relation

$$\mathbf{p}^{a,b}(X, Y) = \mathbf{e}^{a,b}(X, Y) + \mathbf{e}^{b,a}(Y, X) + \delta_{a,b} \frac{\mathbf{e}^{a}(X) - \mathbf{e}^{a}(Y)}{X - Y} \quad (6.10)$$

$$= \mathbf{e}^{a+b}(X + Y, Y) + \mathbf{e}^{a+b,a}(X + Y, X). \quad (6.11)$$

We first calculate the generating functions of the above-defined power series. Set

$$\tilde{g}^{a}(X) = \sum_{k=1}^{\infty} \tilde{g}_{k}^{a}(q)X^{k-1} = -\frac{1}{N} \sum_{n=1}^{\infty} \eta^{an}q^{n}e^{-(nX/N)} \left( 1 - q^{n} \right),$$

$$(\tilde{g}^{a})'(X) = \sum_{k=1}^{\infty} \tilde{g}_{k}^{a}(q)X^{k-1} = \frac{1}{N} \left( \sum_{n=1}^{\infty} \eta^{an}q^{n}e^{-(nX/N)} \left( \frac{q^{n}}{1 - q^{n}} \right)^{2} - \frac{N^{2}f_{2}^{a}(q)}{f_{2}^{a}(q)} \right),$$

$$\tilde{g}^{a,b}(X, Y) = \sum_{r,s=1}^{\infty} \tilde{g}_{r,s}^{a,b}(q)X^{r-1}Y^{s-1} = \frac{1}{N^{2}} \sum_{m,n=1}^{\infty} \eta^{am+bn}q^{m+\frac{1}{2}N} \left( \frac{q^{m}}{1 - q^{m}} \right)^{2} \left( \frac{q^{m+\frac{1}{2}N}}{1 - q^{m+\frac{1}{2}N}} \right),$$

$$\beta^{a,b}(X, Y) = \sum_{r,s=1}^{\infty} \beta_{r,s}^{a,b}(q)X^{r-1}Y^{s-1} = (\tilde{g}^{b}(Y) - \tilde{g}^{a}(X))\beta^{a,b}(X - Y) + \tilde{g}^{a}(X)\beta^{b}(Y),$$

$$\varepsilon^{a,b}(X, Y) = \sum_{r,s=1}^{\infty} \varepsilon_{r,s}^{a,b}(q)X^{r-1}Y^{s-1} = \frac{1}{N} \sum_{n=1}^{\infty} \eta^{an}q^{n} = Nf_{2}^{a}(q).$$

Remark 6.4. (i) Note that $f_{2}^{a}(q) = \tilde{g}_{2}^{a}(q)$ but in general $f_{2}^{a}(q) \neq \tilde{g}_{2}^{a}(q)$.

(ii) Note also that

$$(\tilde{g}_{0}^{a})'(q) = -\sum_{n=1}^{\infty} n\tilde{\eta}_{1}^{a}(q^{n}) = \frac{1}{N} \sum_{n=1}^{\infty} n \sum_{u=1}^{\infty} \eta^{au}q^{un} = Nf_{2}^{a}(q).$$

Turning back to the proof of Theorem 6.3, by the definition, we have

$$\beta^{a}(X) = \sum_{k=1}^{\infty} \beta_{k}^{a}X^{k-1} = -\frac{1}{2NX} \sum_{l=1}^{N} \sum_{n=1}^{\infty} \frac{X^{n}}{n!}e^{-2\pi lai/N} \mathcal{B}_{n} \left( \frac{l}{N} \right)$$

$$= -\frac{1}{2NX} \sum_{l=1}^{N} e^{-2\pi lai/N} \left( \frac{Xe^{X/N}}{Xe^{X/N} - 1} - 1 \right)$$

$$= -\frac{1}{2N} \sum_{l=1}^{N} e^{(X-2\pi ai)/N} \frac{1}{e^{X} - 1} + \delta_{a,0} \frac{e^{-X}}{2X} + \frac{1}{2N} \frac{1}{e^{X} - 1} + \delta_{a,0} \frac{e^{-X}}{2X}.$$
Then
\[ \mathcal{E}^a(X) = \tilde{g}^a(X) - X \tilde{g}_2^a(q) - \tilde{g}_1^a(q), \]
\[ \mathcal{E}^{a,b}(X,Y) = \tilde{g}^{a,b}(X,Y) + \beta^{a,b}(X,Y) + \frac{1}{2N} \varepsilon^{a,b}(X,Y), \]
\[ \mathcal{F}^{a,b}(X,Y) = \tilde{g}^a(X)g^b(Y) + \beta^a(X)\tilde{g}^b(Y) + \beta^b(Y)\tilde{g}^a(X) \]
\[ + \frac{1}{2N}(X(\tilde{g}^b)'(Y) + Y(\tilde{g}^a)'(X)) + M_N \delta_{a,b} (\delta_{a,0} - 1) f_2^a(q). \]

Now we compute \( \mathcal{E}^{a,b}(X,Y) + \mathcal{E}^{b,a}(Y,X) \). Straightforward computation yields
\[
\begin{align*}
\tilde{g}^{a,b}(X,Y) + \tilde{g}^{b,a}(Y,X) &= \frac{\tilde{g}^a(X)g^b(Y) - \tilde{g}^a(X) + \tilde{g}^b(Y)}{2N} \\
&= \frac{1}{2N} \coth \left( \frac{X - Y - 2\pi i(a - b)}{2N} \right) (\tilde{g}^a(X) - \tilde{g}^b(Y)), \quad (6.12) \\
\varepsilon^{a,b}(X,Y) + \varepsilon^{b,a}(Y,X) &= X(\tilde{g}^{b,a})'(Y) + Y(\tilde{g}^{a,b})'(X) + \tilde{g}^a(X) + \tilde{g}^b(Y) \\
&+ N^2 \gamma_{a,b}^N(q) + N^2 \gamma_{b,a}^N(q). \quad (6.13)
\end{align*}
\]

On the other hand, if we set \( \theta = (X - Y - 2\pi(a - b)i)/N \), then
\[
\begin{align*}
\beta^{a,b}(X,Y) + \beta^{b,a}(X,Y) &= (\tilde{g}^b(Y) - \tilde{g}^a(X))\tilde{g}^{a,b}(X - Y) + \tilde{g}^a(X)\beta^b(Y) \\
&- (\tilde{g}^b(Y) - \tilde{g}^a(X))\beta^{b,a}(Y - X) + \tilde{g}^b(Y)\tilde{g}^a(X) \\
&= \frac{\tilde{g}^b(Y) - \tilde{g}^a(X)}{2N} \left( \frac{1}{e^{-\theta} - 1} - \frac{1}{e^\theta - 1} \right) \\
&+ \tilde{g}^a(X)\beta^b(Y) + \tilde{g}^b(Y)\beta^a(X) - \delta_{a,b} \frac{\tilde{g}^a(X) - \tilde{g}^a(Y)}{X - Y} \\
&= \frac{\tilde{g}^a(X) - \tilde{g}^b(Y)}{2N} \coth \left( \frac{X - Y - 2\pi i(a - b)}{2N} \right) \\
&+ \tilde{g}^a(X)\beta^b(Y) + \tilde{g}^b(Y)\beta^a(X) - \delta_{a,b} \frac{\tilde{g}^a(X) - \tilde{g}^a(Y)}{X - Y}. \quad (6.14)
\end{align*}
\]

Adding up (6.12), 1/(2N) \times (6.13) and (6.14), we can derive (6.11) quickly if the conditions in (6.6) are satisfied. Similarly,
\[
\begin{align*}
\tilde{g}^{a+b,a}(X + Y, X) + \tilde{g}^{a+b,b}(X + Y, Y) \\
&= \frac{1}{N^2} \sum_{m \neq n \geq 1} \eta^{m+b+n} e^{-(mX - nY)/N} \frac{q^m}{1 - q^m} \frac{q^n}{1 - q^n} \\
&= \tilde{g}^a(X)\tilde{g}^b(Y) \left( \frac{q^n}{(1 - q^n)^2} - \frac{q^n}{1 - q^n} \right) \\
&= X + Y \frac{\tilde{g}^{a+b,b}(X + Y, X) - f_2^{a+b}(q)}{N} + \frac{1}{N} \tilde{g}^{a+b}(X + Y), \quad (6.15)
\end{align*}
\]

and
\[
\begin{align*}
\varepsilon^{a+b,a}(X + Y, X) + \varepsilon^{a+b,b}(X + Y, Y) \\
&= X(\tilde{g}^{b,a})'(Y) + Y(\tilde{g}^{a,b})'(X) + 2(\tilde{g}^{a+b,b}(X + Y) + 2\tilde{g}^{a+b}(X + Y) \\
&+ 2N f_2^{a+b}(q) - N f_2^a(q) - N f_2^b(q) + N \gamma_{a,b}^N(q) + N \gamma_{b,a}^N(q). \quad (6.16)
\end{align*}
\]
Further,
\[
\beta^{a+b,a}(X + Y, X) + \beta^{a+b,b}(X + Y, Y) \\
= (\tilde{g}^b(Y) - \tilde{g}^{a+b}(X + Y))\beta^a(X) + \tilde{g}^{a+b}(X + Y)\beta^b(Y) \\
+ (\tilde{g}^a(X) - \tilde{g}^{a+b}(X + Y))\beta^b(Y) + \tilde{g}^{a+b}(X + Y)\beta^a(X) \\
= \tilde{g}^b(Y)\beta^a(X) + \tilde{g}^a(X)\beta^b(Y). 
\]

Adding up (6.15), \(1/(2N) \times (6.16)\) and (6.17), we can prove (6.11) if the conditions in (6.7) are satisfied.

To complete the proof of the theorem, we now need to show that system (6.6) together with (6.7) has at least one set of solutions of \(\gamma^{a,b}_N, \lambda^{a,b}_N\) \((0 \leq a, b < N)\) in terms of \(f^a_2(q)\) and \(\bar{g}^a_2(q)\) \((0 \leq a < N)\). Essentially, as a linear algebra problem, this will be solved in Theorem 8.1 in the final section of this paper. This completes the proof of the theorem.

\[\square\]

Let \(\tilde{\zeta}^{a,b}_N(r) = (2\pi i)^{-r} \zeta^{a,b}_N(r), \tilde{\zeta}^{a,b}_N(r, s) = (2\pi i)^{-r-s} \zeta^{a,b}_N(r, s),\) and define \(\tilde{G}^{a}_{r,s}(q)\) and \(\tilde{G}^{a,b}_{r,s;i}(q)\) similarly.

**Theorem 6.5.** Let \(N\) be a positive integer. Let \(z = m \text{ or } s\). Then, for all \(a, b \in \mathbb{Z}/N\mathbb{Z}\) and \(r, s \geq 1\) with \((r, s) \neq 1\), we have
\[
\tilde{G}^{a}_{r,z}(q)\tilde{G}^{b}_{s,z}(q) = \tilde{G}^{a,b}_{r,s; z}(q) + \tilde{G}^{b,a}_{s,z; r}(q) + \delta_{a,b}\tilde{G}^{a}_{r,s; z}(q) + \frac{\delta_{r,1}\tilde{g}^{a}_{s; z}(q) + \delta_{s,1}\tilde{g}^{b}_{r; z}(q)}{2N} \quad (6.18)
\]
where \(f^{a,b}_{r,s}(q) = \left(k^{-2} \right) (\tilde{g}^{a+b}_{k-1})'(q) + \tilde{g}^{a+b}_{k-1}(q))/N\) with \(k = r + s\). Moreover,
\[
\tilde{G}^{a}_{1,z}(q)\tilde{G}^{b}_{1,z}(q) = \tilde{G}^{a+b}_{1,1; z}(q) + \tilde{G}^{a+b,a}_{1,1; z}(q) + \frac{1}{2}(f^{a}_{2} + f^{b}_{2}) \\
+ \frac{1}{2N}(2\tilde{g}^{a+b}_{0} + 2\tilde{g}^{a+b}_{0} - (\tilde{g}^{a}_{0})' + (\tilde{g}^{b}_{0})'). \quad (6.20)
\]

**Proof.** By Corollary 5.4 and (5.6), we have
\[
\tilde{\beta}^{a} + \beta^{b} = \tilde{\zeta}^{a}_N(s).
\]
So by the definitions,
\[
\tilde{G}^{a}_{r,z}(q) = \tilde{\zeta}^{a}_N(r) + \tilde{g}^{a}_{r}(q), \\
\tilde{G}^{a,b}_{r,s; z}(q) = \tilde{\zeta}^{a,b}_N(r, s) + \tilde{g}^{a,b}_{r,s}(q) + \beta^{a,b}_{r,s}(q). \quad (6.21)
\]

Thus
\[
\tilde{G}^{a}_{r,z}(q)\tilde{G}^{b}_{s,z}(q) + \frac{\delta_{r,2}(\tilde{g}^{a}_{0})' + \delta_{s,2}(\tilde{g}^{b}_{0})'}{2N} \\
= \tilde{\zeta}^{a}_N(r)\tilde{\zeta}^{b}_N(s) + P^{a,b}_{r,s}(q) + \tilde{\beta}^{a}\tilde{g}^{b}_{r}(q) + \tilde{\beta}^{b}\tilde{g}^{a}_{r}(q) \\
= \tilde{\zeta}^{a,b}_N(r, s) + E^{a,b}_{r,s}(q) + \tilde{\zeta}^{a,b}_N(r, s) + E^{a,b}_{r,s}(q) + \delta_{a,b}(\tilde{\zeta}^{a}_N(r + s) + E^{a}_r(q)) + I^{a,b}_{r,s}(q) + I^{b,a}_{r,s}(q) \\
= \sum_{i+j=r+s, i,j \geq 1} \left( (i - 1) \left( \tilde{\zeta}^{a+b,b}_{r,s; z}(i, j) + E^{a+b,b}_{i,j}(q) + I^{a+b,b}_{i,j}(q) \right) + (j - 1) \left( \tilde{\zeta}^{a+a,b}_{r,s; z}(i, j) + E^{a+a,b}_{i,j}(q) + I^{a+a,b}_{i,j}(q) \right) \right) \\
+ \left( i - 1 \right) \left( \tilde{\zeta}^{a,b,a}_{r,s; z}(i, j) + E^{a+b,a}_{i,j}(q) + I^{a+b,a}_{i,j}(q) \right) \quad (6.22)
\]
by Propositions 3.2, 5.6 and Theorem 6.3. Note that \((r, s) \neq (1, 1)\) just because \(r + s \geq 3\) when using Proposition 5.6. Now by the definition
\[
\varepsilon_{r,s}^{a,b}(q) + \varepsilon_{s,r}^{b,a}(q) = \delta_{s,2}(\bar{g}^a_r)'(q) + \delta_{r,2}(\bar{g}^b_s)'(q) + \delta_{s,1}\bar{g}^a_r(q) + \delta_{r,1}\bar{g}^b_s(q).
\]
Hence (6.18) follows from Definition 6.2, (6.21) and (6.22). For (6.19), we need to compute
\[
\sum_{i+j=r+s} \left[ \left( \frac{i-1}{r-1} \right) \varepsilon_{i,j}^{a+b,b}(q) + \left( \frac{i-1}{s-1} \right) \varepsilon_{i,j}^{a+b,a}(q) \right]
\]
\[
= \sum_{i+j=r+s} \left[ \left( \frac{i-1}{r-1} \right) (\delta_{i,2}(\bar{g}^b_j)'(q) - \delta_{i,1}(\bar{g}^b_j-1)'(q) + \delta_{j,1}(\bar{g}^{a+b}_i)'(q) + \bar{g}^{a+b}_i(q)) \right]
\]
\[
+ \left( \frac{i-1}{s-1} \right) (\delta_{i,2}(\bar{g}^a_i)'(q) - \delta_{i,1}(\bar{g}^a_i-1)'(q) + \delta_{j,1}(\bar{g}^{a+b}_j)'(q) + \bar{g}^{a+b}_j(q)) \right]
\]
\[
= \delta_{r,2}(\bar{g}^b_s)'(q) + \delta_{s,2}(\bar{g}^a_i)'(q) + \left[ \left( \frac{k-2}{r-1} \right) + \left( \frac{k-2}{s-1} \right) \right] ((\bar{g}^{a+b}_{k-2})'(q) + \bar{g}^{a+b}_{k-1}(q)),
\]
where \(k = r + s\). This yields (6.22) immediately.

Finally, (6.20) follows from direct computation using (5.8) and (6.7). This completes the proof of the theorem.

\[ \square \]

**Remark 6.6.** When \(N = 1\), Theorem 6.5 reduces to [7, Theorem 7]. When \(N = 2\), Theorem 6.5 reduces to [9, Theorem 3] with some correction there.

### 7. A key relation on multiple divisor functions at level \(N\)

In this section, we prove a key result on multiple divisor functions at level \(N\), which will be used in the next section.

Let \(\varphi\) be Euler’s totient function. We first need a lemma concerning some special power sums of roots of unity.

**Lemma 7.1.** Let \(N = \prod_{t=1}^r p_i^{k_t}\) and \(\eta\) be a primitive \(N\)th root of unity. For \(\alpha_t \leq k_t\), \(t = 1, \ldots, r\) (but \(\alpha = (\alpha_1, \ldots, \alpha_r) \neq (k_1, \ldots, k_r)\)) we write
\[
J(\alpha) = J_N(\alpha_1, \ldots, \alpha_r) = \{1 \leq i < N : p_i^{\alpha_t} || i \quad \forall t = 1, \ldots, r\}.
\]

Then, for any choice of \(r\)-tuple of non-negative integers \((\ell_1, \ldots, \ell_r)\) we have
\[
\sum_{i \in J(\alpha_1, \ldots, \alpha_s)} 0^i \prod_{t=1}^r p_i^{\ell_t} = \prod_{t \in I} (-p_i^{\ell_t}) \prod_{s \notin I} \varphi(p_i^{k_t-\alpha_t}) \quad \text{if } C_I \text{ holds,}
\]
where \(C_I\) is the condition that there is \(I \subseteq \{1, \ldots, r\}\) such that \(\ell_t = k_t - \alpha_t - 1\) for all \(t \in I\) and \(\ell_s \geq k_s - \alpha_s\) for all \(s \notin I\).

**Proof.** Suppose that \(\xi\) is a primitive \(p^k\)th root of unity for some prime \(p\). Then, for all \(\alpha < k\) we have
\[
\sum_{p^\alpha || i, 1 \leq i < p^k} \xi^i = \sum_{p^\alpha || i, 1 \leq i < p^k} \xi^i - \sum_{p^{\alpha+1} || i, 1 \leq i < p^k} \xi^i = \begin{cases} -1 & \text{if } \alpha = k - 1, \\ 0 & \text{if } \alpha < k - 1, \end{cases}
\]
(7.1)
since, for any divisor $D$ of $p^k$, we have

$$\sum_{D | i, 1 \leq i < p^k} \xi^i = \begin{cases} -1 & \text{if } D < p^k, \\ 0 & \text{if } D = p^k. \end{cases}$$

Let $N_t = p_t^{k_t}$ for all $t = 1, \ldots, r$. It is well known that $\eta$ can be decomposed as $\eta = \prod_{t=1}^r \xi_t^i$, where $\xi_t^i$ is a primitive $N_t$th root of unity for each $t$. Then $\xi_t = (\xi_t^i)^{r_{t-1}}$, where $r_{t-1}$ is a primitive $N_t$th root of unity. By the Chinese Remainder Theorem, it is easy to see that

$$\sum_{i \in J_{\alpha_1, \ldots, \alpha_r}} \eta^i \prod_{t=1}^r p_t^{k_t} = \prod_{t=1}^r \left( \sum_{i \in J_{\alpha_t}(\alpha_t)} \xi_t^{i} p_t^{k_t} \right).$$

The lemma now follows from (7.1) and the fact that $|J_{\alpha_t}(\alpha_t)| = \varphi(p_t^{k_t-\alpha_t}).$

Recall that, for a $a \in \mathbb{Z}/N\mathbb{Z}$, we have defined the level $N$ divisor functions $\sigma_1^N(m) = \sum_{nu=m} \eta^{au} u$ and $\kappa_1^N(m) = \sum_{nu=m} \eta^{an} u$.

**Theorem 7.2.** Let $N = \prod_{t=1}^r p_t^{k_t}$, where $p_1, \ldots, p_r$ are pairwise distinct prime factors of $N$. Then, for all $m \in \mathbb{N}$, we have

$$\sum_{\gcd(N,i)=1, 1 \leq i < N} \sigma_1^N(m) - \varphi(N) \sigma_1^N(m)$$

$$= \sum_{i \in \{1, \ldots, r\}} \left( \prod_{i \in I} \varphi(p_i^{k_i}) \prod_{p_i | i} \prod_{p_i | i} \sum_{\gcd(i, p_i) = 1} \kappa_1^N(m) \right). \quad (7.2)$$

**Proof.** To save space, we put $|r| = \{1, \ldots, r\}$. Let $e_t \geq 0$ for all $t \in [r]$ and assume $m = \prod_{t=1}^r p_t^{e_t} p_{r+1}^{e_{r+1}}$, where $p_1, \ldots, p_r$ are pairwise distinct primes. Set $Q = \prod_{t=r+1}^r (1 + p_t + \cdots + p_t^{e_t})$ ($Q = 1$ if none of $p_{r+1}, \ldots, p_r$ appears).

If $\ell_t \leq e_t$ for all $t \in [r]$, then

$$\sum_{\nu u = m, p_t^{e_t} \mid u, \forall t \in [r]} \sum_{i \in J_{\alpha_1, \ldots, \alpha_r}} \eta^i \nu u = \prod_{t=1}^r p_t^{e_t} \sum_{i \in J_{\alpha_1, \ldots, \alpha_r}} \eta^i \prod_{t=1}^r p_t^{e_t}.$$

If $\alpha_t > k_t$ for some $t \in [r]$, then $\sum_{i \in J_{\alpha_1, \ldots, \alpha_r}} \sigma_1^N(m) = 0$ by the definition of $J$. For any partition $[r] = \Pi \Lambda = \Lambda_1 \Pi \Lambda_2 \Pi \Lambda_3$ with $\Lambda_1 = (\Lambda_1, \Lambda_2, \Lambda_3) \neq (\emptyset, \emptyset, [r])$ we write $\alpha = (\alpha_1, \ldots, \alpha_r) \mapsto \Lambda$ if $\alpha_t = 0$ for all $t \in \Lambda_1$, $1 \leq \alpha_t < k_t$ for all $t \in \Lambda_2$ and $\alpha_t = k_t$ for all $t \in \Lambda_3$. We remove the case $\Lambda = (\emptyset, \emptyset, [r])$ since $(\alpha_1, \ldots, \alpha_r) \neq (k_1, \ldots, k_r)$. For such $\alpha$ we have, by Lemma 7.1,

$$\sum_{i \in J_{\alpha_1, \ldots, \alpha_r}} \kappa_1^N(m) = Q \prod_{t \in \Lambda_3} \left( -p_t^{e_t} + \varphi(p_t^{k_t-\alpha_t}) \sum_{\ell_t = k_t-\alpha_t}^{e_t} \frac{p_t^{\ell_t} - 1}{p_t - 1} \right) \prod_{t \in \Lambda_3} \left( \sum_{\ell_t = 0}^{e_t} p_t^{\ell_t} - 1 \right)$$

$$= Q \prod_{t \in \Lambda_3} \left( -p_t^{k_t-\alpha_t-1} \sum_{\ell_t = k_t-\alpha_t}^{e_t} \frac{p_t^{\ell_t+1} - 1}{p_t - 1} \right),$$

$$\sum_{i \in J_{\alpha_1, \ldots, \alpha_r}} \sigma_1^N(m) = Q \prod_{t \in \Lambda_3} \left( p_t^{2(k_t-\alpha_t-1)} + \varphi(p_t^{k_t-\alpha_t}) \sum_{\ell_t = k_t-\alpha_t}^{e_t} p_t^{\ell_t} \right) \prod_{t \in \Lambda_3} \left( \sum_{\ell_t = 0}^{e_t} p_t^{\ell_t} \right)$$

$$= Q \prod_{t \in \Lambda_3} \left( p_t^{k_t-\alpha_t-1}(p_t^{e_t+1} - p_t^{k_t-\alpha_t} - p_t^{k_t-\alpha_t-1}) \prod_{t \in \Lambda_3} \left( \sum_{\ell_t = 0}^{e_t} p_t^{\ell_t} \right) \right) \prod_{t \in \Lambda_3} \left( \sum_{\ell_t = 0}^{e_t} p_t^{\ell_t} \right)$$

$$= Q \prod_{t \in \Lambda_3} \left( p_t^{k_t-\alpha_t-1}(p_t^{e_t+1} - p_t^{k_t-\alpha_t} - p_t^{k_t-\alpha_t-1}) \prod_{t \in \Lambda_3} \left( \sum_{\ell_t = 0}^{e_t} p_t^{\ell_t} \right) \right) \prod_{t \in \Lambda_3} \left( \sum_{\ell_t = 0}^{e_t} p_t^{\ell_t} \right).$$
if \( e_t \geq k_t - 1 \) for all \( t \in [r] \). If \( e_t < k_t - 1 \), then
\[
\sum_{i \in J(\alpha_1, \ldots, \alpha_r)} \kappa^i_1(m) = \sum_{i \in J(\alpha_1, \ldots, \alpha_r)} \sigma^i_1(m) = 0
\]
when \( \alpha_t < k_t - e_t - 1 \) for some \( t \in [r] \). Therefore, we have
\[
\sum_{i \in J(0, \ldots, 0)} \sigma^i_1(m) = \begin{cases} 0 & \text{if } e_t < k_t - 1 \text{ for some } t \in [r], \\ \prod_{t \in \Lambda_1} \phi(p_t^{k_t}) \prod_{s \not\in I} \sum_{p_t | i \prod_{t \in \Lambda} \phi(p_t^{k_t})(p_t^{e_t+1} - p_t^{e_t} - 1))} & \text{otherwise.} \end{cases}
\]

Now we write \( \sum'_{\Pi \Lambda = [r]} \) to mean that in the sum \( \Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \) runs through all partitions of \([r]\) into three parts except for \((0, 0, [r])\). Then
\[
\sum_{I \subseteq \{1, \ldots, r\}} \left( \prod_{t \in I} \phi(p_t^{k_t}) \prod_{s \not\in I} \sum_{p_t | i \prod_{t \in \Lambda} \phi(p_t^{k_t})(p_t^{e_t+1} - p_t^{e_t} - 1))} + \phi(p_t^{k_t})(p_t^{e_t+1} - 1) \right)
\]
\[
= Q \prod_{t=1}^r F_t - Q \prod_{t=1}^r \phi(p_t^{k_t})(p_t^{e_t+1} - 1) 
\]
\[
= Q \prod_{t=1}^r F_t - \varphi(N) \sigma^0_1(m)
\]
where if \( e_t < k_t - 1 \), then
\[
F_t = - \left( \sum_{\alpha_t = k_t - e_t - 1}^{k_t - 1} \phi(p_t^{k_t})p_t^{k_t - \alpha_t - 1} \right) + \phi(p_t^{k_t})(p_t^{e_t+1} - 1) = 0,
\]
and if \( e_t \geq k_t - 1 \), then
\[
F_t = -p_t^{k_t} - \left( \sum_{\alpha_t = 1}^{k_t - 1} \phi(p_t^{k_t})p_t^{k_t - \alpha_t - 1} \right) + \phi(p_t^{k_t})(p_t^{e_t+1} - 1) 
\]
\[
= -p_t^{k_t - 1}(p_t^{e_t+1} - p_t^{k_t} - p_t^{k_t - 1}).
\]
The theorem now follows at once.

**Corollary 7.3.** Let \( N = \prod_{i=1}^r p_i^{k_i} \), where \( p_1, \ldots, p_r \) are pairwise distinct prime factors of \( N \). Then we have

\[
\sum_{\gcd(N, i) = 1, 1 \leq i < N} \tilde{g}_2(q) - \varphi(N)f_2^0(q)
\]
\[
= \sum_{I \subseteq \{1, \ldots, r\}} \left( \prod_{t \in I} \phi(p_t^{k_t}) \prod_{s \not\in I} \sum_{p_t | i \prod_{t \in \Lambda} \phi(p_t^{k_t})(p_t^{e_t+1} - p_t^{e_t} - 1))} f_2^i(q) \right).
\]

**Example 7.4.** Let \( p \) be a prime and the level \( N = p^k \). For any \( m \in \mathbb{N} \), we have
\[
\sum_{p | i, 1 \leq i < N} \tilde{g}_2(q) - \varphi(N)f_2^0(q) = \varphi(N) \sum_{p | i, 1 \leq i < N} f_2^i(q) + N \sum_{p | i, 1 \leq i < N} f_2^i(q).
Example 7.5. Let $p_1$ and $p_2$ be two distinct prime numbers and $N = p_1^j p_2^k$. Then we have
\[
\sum_{p_1^j t, p_2^k | t, 1 \leq i < N} \tilde{g}_2^j(q) - \varphi(N) f_2^j(q) = \varphi(N) \sum_{p_1^j t, p_2^k | t, 1 \leq i < N} f_2^j(q)
+ p_1^j \varphi(p_2^k) \sum_{p_1^j t, p_2^k | t, 1 \leq i < N} f_2^j(q) + \varphi(p_1^j) p_2^k \sum_{p_1^j t, p_2^k | t, 1 \leq i < N} f_2^j(q)
+ N \sum_{p_1^j t, p_2^k | t, 1 \leq i < N} f_2^j(q).
\]

8. A linear algebra problem

In this section, using the standard techniques from linear algebra and the key result on the multiple divisor functions at level $N$ proved in the preceding section, we will derive the solvability of a system of linear equations associated with (6.6) and (6.7) for every positive integer $N$. This completes the proof of our main result on the level-$N$ Eisenstein series given in Theorem 6.3.

For every positive integer $N$ we let $\nu(N)$ be the number of its positive divisors (including 1 and $N$ itself).

Theorem 8.1. For every positive integer $N$, system (6.6) together with (6.7) has infinitely many sets of solutions of $\gamma_N^{a,b}$ and $\lambda_N^{a,b} = \lambda_N^{a,a} \lambda_N^{b,b} (0 \leq a, b < N)$ in terms of $f_2^j(q)$ and $\tilde{g}_2^j(q) (0 \leq a < N)$. Moreover, one can always choose
\[
\{\gamma_N^{a,b} : 0 \leq b < a < N\} \cup \{\gamma_N^{N-a,N-a} : 1 \leq a \leq N, a | N\}
\]
(8.1)
as the $N(N - 1)/2 + \nu(N)$ free variables.

Before giving its proof, we first analyze the linear system in Theorem 8.1 using standard techniques from linear algebra. Let $x_N$ be a column vector with $(3N^2 + N)/2$ components whose transpose is
\[
tx_N = (\gamma_N^{0,0}, \gamma_N^{0,1}, \gamma_N^{0,2}, \ldots, \gamma_N^{N-1,N-1}, \gamma_N^{0,1}, \gamma_N^{0,2}, \gamma_N^{0,3}, \ldots, \gamma_N^{N-2,N-1},
\gamma_N^{1,0}, \gamma_N^{1,1}, \gamma_N^{1,2}, \gamma_N^{2,2}, \gamma_N^{3,2}, \ldots, \gamma_N^{N-1,1}, \gamma_N^{0,1}, \gamma_N^{0,2}, \gamma_N^{0,3}, \ldots, \gamma_N^{N-2,N-1}).
\]

Here the rule to list the entries is to use lexicographic order for $\lambda_N^{a,b} (0 \leq a < b < N)$, then $\gamma_N^{a,b}$ ($0 \leq a < b < N$), then $\gamma_N^{a,a} (0 \leq a < N)$ and finally $\gamma_N^{a,b}$ ($0 \leq b < a < N$). Then we can rewrite system (6.6) together with (6.7) as follows:
\[
\begin{align*}
\gamma_N^{a+b,0} + & \gamma_N^{a+b,a} - 2 \lambda_N^{a,b} = f_2^a + f_2^b & \forall 0 \leq a \leq b < N, \quad (\text{LS}_1^{a,b}) \\
\gamma_N^{a,b} + & \gamma_N^{b,a} - \gamma_N^{a+h,a} - \gamma_N^{a+h,b} = -f_2^a - f_2^b & \forall 0 \leq a \leq b < N, \quad (\text{LS}_2^{a,b}) \\
\gamma_N^{a,a} - & \gamma_N^{2a,a} = q_2^a - f_2^a & \forall 1 \leq a < N, \quad (\text{LS}_3^{a})
\end{align*}
\]
where the last two families of equations are obtained by taking the difference of (6.6) and (6.7).

We then can express this system by a single matrix equation
\[
A_N x_N = b_N
\]
(8.2)
for some matrix $A_N$ of size $(N^2 + N - 1) \times (3N^2 + N)/2$ and a column vector $b_N$ of length $N^2 + N$ whose entries are given in terms of only the quantities $f_2^a$ and $\tilde{g}_2^a$ only. Note that since $(\text{LS}_3^a)$ is trivial, the row size is decreased from $N^2 + N$ by 1. To prove the proposition, one thing we need to show is that every row vector in the left null space $\mathcal{N}(A_N)$ of $A_N$ annihilates $b_N$. 
Example 8.2. When $N = 1$, we get the equation
\[
\begin{bmatrix} -2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1^{0,0} \\ \gamma_1^{0,0} \end{bmatrix} = 2f_2^0.
\]
Clearly, $\mathcal{N}(A_1) = \emptyset$ and we may choose $\gamma_1^{0,0}$ arbitrarily and then set $\lambda_1^{0,0} = \gamma_1^{0,0} = f_2^0$.

Example 8.3. When $N = 2$, we get the equation
\[
A_2 \mathbf{x}_2 = \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_2^{0,0} \\ \lambda_2^{0,1} \\ \lambda_2^{1,0} \\ \lambda_2^{1,1} \\ \gamma_2^{0,0} \\ \gamma_2^{0,1} \\ \gamma_2^{1,0} \\ \gamma_2^{1,1} \end{bmatrix} = \begin{bmatrix} f_2^0 \\ f_2^0 + f_2^1 \\ f_2^1 \\ f_2^1 + \tilde{\gamma}_2^1 - f_2^2 \\ -f_2^0 - f_2^1 \end{bmatrix} = \mathbf{b}_2. \quad (8.3)
\]

Then $\mathcal{N}(A_2)$ is spanned by the vector $\mathbf{n}_2 = (0, 0, 0, 0, 1, 1)$. We see that
\[
\mathbf{n}_2 \cdot \mathbf{b}_2 = f_2^0 + 2f_1^1 - \tilde{\gamma}_2^1 = 0
\]
which follows from Example 7.4 by taking $p = 2$ and $k = 1$ there. This implies that system (8.3) has infinitely many solutions. Setting $\gamma_2^{a,b} = 0$ for $1 \geq a \geq b \geq 0$ (in fact, one may choose them arbitrarily as they are free variables), we need to solve only the system
\[
A'_2 \mathbf{x}_2 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_2^{0,0} \\ \lambda_2^{0,1} \\ \lambda_2^{1,1} \\ \gamma_2^{1,0} \end{bmatrix} = \begin{bmatrix} f_2^0 \\ f_2^0 + f_2^1 \\ f_2^1 \\ -f_2^0 - f_2^1 \end{bmatrix} = \mathbf{b}'_2.
\]
Here we obtain $A'_2$ from $A_2$ by removing the penultimate row of $A_2$ (which is equivalent to removing the equation $\gamma_N^{0,1} = \gamma_N^{1,0} = \tilde{\gamma}_2^1 - f_2^2$), and then removing the last three columns (which is equivalent to setting $\gamma_2^{a,b} = 0$ for $1 \geq a \geq b \geq 0$). Correspondingly, we obtain $\mathbf{b}'_2$ by removing the penultimate entry $\tilde{\gamma}_2^1 - f_2^2$ of $\mathbf{b}_2$. Clearly, this new system has a unique solution, which also gives a solution of the original system:
\[
\lambda_2^{0,0} = -f_2^0, \quad \lambda_2^{0,1} = \frac{f_2^1 - \tilde{\gamma}_2^1}{2}, \quad \lambda_2^{1,1} = -\tilde{\gamma}_2^1, \quad \gamma_2^{0,1} = 2\lambda_2^{0,1}, \quad \gamma_2^{a,b} = 0, \quad \forall \ 1 \geq a \geq b \geq 0.
\]

We can also check that (6.2) with (6.4) provides another set of solutions of (8.3).

Remark 8.4. In an email Kaneko and Tasaka pointed out to us that [9, (19)] should be corrected as follows:
\[
\alpha_1 = \tilde{\gamma}_0^0(q), \quad \alpha_2 = -\alpha_1, \quad \alpha_3 = 2\tilde{\gamma}_0^0(q) + \bar{g}_0^0(q).
\]
Together with their choice $\lambda_2^{0,1} = \lambda_2^{1,0} = \lambda_2^{1,1} = 0$ given in [9, Theorem 3], we find the following solution to (8.3):
\[
\gamma_2^{0,0} = f_2^0, \quad \gamma_2^{0,1} = f_2^1, \quad \gamma_2^{1,0} = -f_2^1, \quad \gamma_2^{1,1} = \tilde{\gamma}_2^1, \quad \lambda_2^{0,0} = \lambda_2^{0,1} = \lambda_2^{1,1} = 0,
\]
since we have the following correspondence between their notation and ours:
\[
\alpha_1 \leftrightarrow 2\gamma_2^{0,1}, \quad \alpha_2 \leftrightarrow 2\gamma_2^{1,0}, \quad \alpha_3 \leftrightarrow 2\gamma_2^{1,1}, \quad \tilde{\gamma}_0^0 \leftrightarrow 2f_2^0, \quad \bar{g}_0^0 \leftrightarrow 2f_2^1, \quad \tilde{\gamma}_2^1 \leftrightarrow \bar{g}_2^1.
\]
EXAMPLE 8.5. Similarly, when $N = 3$, we see that $\mathcal{N}(A_3)$ is spanned by the vector $n_3 = (0, 0, 0, 0, 0, 0, 1, 1, 1, 1)$ and
\[
\mathbf{n}_3 \cdot \mathbf{b}_3 = g_2^1 + g_2^2 - 2f_2^0 - 3f_2^1 - 3f_2^2 = 0
\]
which follows from Example 7.4 by taking $p = 3$ and $k = 1$ there. Further, we can obtain $A'_3$ from the $11 \times 15$ matrix $A_3$ by removing the row of $A_3$ corresponding to the equations $\gamma_N^{2,a} - \gamma_N^{a,a} = f_2^1 - g_2^2$ for $a = 2$ and then removing the 10th and the last 4 columns (which is equivalent to setting $\gamma_2^{a,b} = 0$ for all $2 \geq a \geq b \geq 0$ with $(a, b) \neq (1, 1)$). In this way, we find the following solution:
\[
\begin{align*}
\lambda_3^{0,0} &= -f_2^0, & \lambda_3^{0,1} &= \frac{g_2^1 - f_2^0 - 2f_2^0}{2}, & \lambda_3^{1,1} &= -f_2^1, & \lambda_3^{1,2} &= \frac{g_2^2}{2}, \\
\gamma_3^{0,2} &= -\frac{f_2^0 + f_2^2}{2}, & \lambda_3^{2,2} &= 2\lambda_3^{1,2}, & \gamma_3^{1,0} &= g_2^1 - f_2^0 - 2f_2^1, & \gamma_3^{0,2} &= -f_2^0 - f_2^2, \\
\gamma_3^{1,2} &= f_2^2 - g_2^2, & \gamma_3^{1,1} &= g_2^1 - f_2^1, & \gamma_3^{a,b} &= 0, & \forall a \geq b, (a, b) \neq (1, 1).
\end{align*}
\]

Of course, this solution is not unique. For instance, we checked that (6.3) with (6.5) provides another set of solutions of the system $A_3x = b_3$.

For other levels $N \leq 80$, we carried out similar computations using Maple and verified that $\mathcal{N}(A_N)$ always annihilates $b_N$ using Corollary 7.3. To prove the general case, we need two results concerning dimensions.

**Proposition 8.6.** The dimension of the left null space of $\mathcal{N}(A_N)$ satisfies
\[
\text{dim } \mathcal{N}(A_N) \geq \nu(N) - 1.
\]
Moreover, for every vector $\mathbf{n} \in \mathcal{N}(A_N)$ we have $\mathbf{n} \cdot \mathbf{b}_N = 0$.

**Proof.** Throughout this proof, we will drop the subscript $N$. For each divisor $d$ of $N$, if $d < N$, then we can obtain a vector $n_N(d) \in \mathcal{N}(A_N)$ by using the following combination of the families of equations in $(LS_2^{a,b})$ and $(LS_3^{a,b})$:

- Adding $(LS_2^{a,b})$ with $\gcd(b, N) = d$ and $a = 0$,
- Adding $(LS_2^{a,b})$ with $\gcd(a, b, N) = d$ for all $1 \leq a < b < N$ and
- Adding $(LS_3^{a,b})$ with $\gcd(a, N) = d$.

This gives rise to the vector $n_N(d)$ whose entries are either 0 or 1 and whose leading 1 occurs at the position corresponding to the variable $\gamma_0^{a,d}$. We will show $n_N(d) \in \mathcal{N}(A_N)$ and $n_N(d) \cdot b_N = 0$, which is equivalent to the fact that LHS = 0 and RHS = 0, respectively, where

\[
\text{LHS} = \sum_{\gcd(a, N) = d} \sum_{1 \leq a < b < N} (\gamma_0^{a,d} - \gamma_2^{2,a}) + \sum_{\gcd(a, b, N) = d} (\gamma_1^{a,b} + \gamma_2^{b,a} - \gamma_2^{a+b,a} - \gamma_2^{a+b,b})
\]

and

\[
\text{RHS} = \sum_{\gcd(a, N) = d} (\tilde{g}_2 - f_2^0) - \sum_{\gcd(a, N) = d} (f_2^0 + f_2^0) - \sum_{\gcd(a, b, N) = d} (f_2^a + f_2^b).
\]

Also note that the vectors in $\{n_N(d) : d | N, d < N\}$ are linearly independent since the leading 1 appearing in $n_N(d)$ corresponds to $\gamma_0^{a,d}$. This implies $\text{dim } \mathcal{N}(A_N) \geq \nu(N) - 1$.

By considering $(N/d)$th roots of unity (that is, reducing to level $N/d$), we may assume without loss of generality that $d = 1$ and we simply write $\mathbf{n}$ for $n_N(1)$. To show LHS = 0, we
break the two sums in (8.4) into the following parts:

\[(A_n) : \gamma^{0,a} \text{ from first sum,} \]
\[(B_n) : -\gamma^{2a,a}, \text{ first sum,} \]
\[(C_{1a}^b) : \gamma^{a,b} = \gamma^{N+a,a}, \text{ second sum,} \]
\[(C_{2a}^b) : \gamma^{b,a}, \text{ second sum,} \]
\[(D_{1a}^b) : -\gamma^{a+a,a}, \text{ second sum,} \]
\[(D_{2a}^b) : -\gamma^{a+b,b}, \text{ second sum,} \]
\[(E_1^a) : -\gamma^{b,a}, \text{ second sum where } b = N - a \] (so \(a < N/2\)),
\[(E_2^a) : -\gamma^{b,b}, \text{ second sum where } a = N - b \] (so \(b > N/2\)).

Then we obtain the cancelations as follows:

\[
\sum_{\gcd(a,N)=1} \frac{A_a + \sum_{\gcd(a,N)=1, a<N/2} E_1^a + \sum_{\gcd(b,N)=1, b>N/2} E_2^b}{\gcd(a,b,N)=1} = 0,
\]
\[
\sum_{\gcd(a,N)=1} \frac{A_a + \sum_{\gcd(a,N)=1, a<N/2} E_1^a + \sum_{\gcd(b,N)=1, b>N/2} E_2^b}{\gcd(a,b,N)=1} = 0,
\]
\[
\sum_{\gcd(a,b,N)=1, a<b} (C_{1a}^b + C_{2a}^b) = \sum_{\gcd(a,b,N)=1, 2N>\alpha >b>0} \gamma^{a,b} = CC_1 + CC_2 + CC_3,
\]

where

\[
CC_1 = \sum_{\gcd(a,b,N)=1, 2N>\alpha >2b>0} \gamma^{a,b} = -\sum_{\gcd(a,N)=1} B_a,
\]
\[
CC_2 = \sum_{\gcd(a,b,N)=1, 2N>\alpha >2b>0} \gamma^{a,b} = \sum_{\gcd(a,b,N)=1, 2N>\alpha >b>0} \gamma^{(a-b)+b,b} = -\sum_{\gcd(a,b,N)=d, a<b} D_{1a}^b,
\]
\[
CC_3 = \sum_{\gcd(a,b,N)=1, 2N>\alpha >2b>0} \gamma^{a,b} = \sum_{\gcd(a,b,N)=1, 2N>\alpha >b>0} \gamma^{(a-b)+b,b} = -\sum_{\gcd(a,b,N)=d, a<b} D_{2a}^b.
\]

These imply \(LHS = 0\), which shows \(n \in N(A_N)\).

Now we turn to RHS. Since the \(N = 1\) case is trivial, we now assume \(N = \prod_{i=1}^r p_i^{k_i} > 1\), where \(p_1, \ldots, p_r\) are pairwise distinct prime factors of \(N\). Recall that

\[
RHS = \sum_{\gcd(a,N)=d, 1 \leq a < N} (g_2^a - f_2^a) - \sum_{\gcd(a,N)=d, 1 \leq a < N} (f_2^a + f_2^b) - \sum_{\gcd(a,N)=d, 1 \leq a < b < N} (f_2^a + f_2^b). \tag{8.5}
\]

We want to show that the above expression is exactly equal to the difference of the two sides in Corollary 7.3, which is therefore 0. Clearly, the coefficients of \(g_2^a \) (1) and \(f_2^b \) (\(= \varphi(N)\)) are correct.

Let \(\gcd(c, N) = 1\). Now we count how many times \(f_2^c\) can appear. Note that if \(a \neq c\), then we have \(\gcd(a, c, N) = 1\) and either \(a < c\) or \(a > c\). Thus the last sum in (8.5) contributes \(N - 1\) copies of \(f_2^c\). Combining this with the one copy from the sum \(\sum_{\gcd(a,N)=d} (f_2^0 + f_2^a)\), we see that the coefficient of \(f_2^c\) is exactly \(N\).

Now we consider \(f_2^c\) with \(\gcd(c, N) > 1\). Without loss of generality, we assume that there is \(1 \leq t < r\) such that \(p_i \mid c\) for all \(i \leq t\) and \(p_j \mid c\) for all \(t < j \leq r\). Then only the last sum in (8.5) has non-trivial contributions. In fact, for 1 \(\leq a < N\) we see that \(p_i \mid a\) \((i = 1, \ldots, t)\) if and only if \(\gcd(a, c, N) = 1\). But obviously the number of such \(a\) is given by

\[
\prod_{i=1}^t \varphi(p_i^{k_i}) \prod_{j=t+1}^r p_j^{k_j}
\]

which agrees with Corollary 7.3. This implies \(RHS = 0\), which shows \(n \cdot b_N = 0\).

We have completed the proof of the lemma.
\[\square\]
Proposition 8.7. The rank of matrix $A_N$ satisfies
\[ \text{rank}(A_N) \geq N^2 + N - \nu(N). \]
Moreover, one may choose the free variables as in (8.1) when solving the linear system (6.6) + (6.7) (or, equivalently, the linear system $(\text{LS}_1^{a,b}) + (\text{LS}_2^{a,b}) + (\text{LS}_3^a)$).

Proof. We will drop the subscript $N$ again for the variables $\lambda$ and $\gamma$. We will prove the lemma by producing the following pivot variables in $x_N$:
\[ S = \{ \lambda^{a,b} : 0 \leq a \leq b < N \} \cup \{ \gamma^{a,b} : 0 \leq a < b < N \} \cup \{ \gamma^{a,a} : 1 \leq a < N, (N-a) \nmid N \}. \]  

(8.6)

Easy computation shows $|S| = N^2 + N - \nu(N)$, which yields the lemma immediately.

To streamline our proof, we start with some ad hoc terminology. Suppose that we have a linear system of variables $x_1, \ldots, x_r$ (in this particular order). Then an equation produced from this system by the elementary operations (namely, multiplying an equation by a scalar and adding or subtracting two equations) is called a pivotal equation of variable $x_i$ if $x_i$ appears in the equation while none of $x_1, \ldots, x_{i-1}$ does. In our situation, our variables $\lambda$ and $\gamma$ are ordered as in the vector $x_N$. And clearly $(\text{LS}_{a,b}^1)$ provides the pivotal equations of the variables $\lambda^{a,b}$.

We now turn to the variables $\gamma^{a,b}$. We shall produce their pivotal equations by the following steps. We write $\gamma^{a,b} = \cdots$ to mean that the right-hand side does not involve any variables from $S$. In particular, we may omit all $\gamma^{a,b}$ with $a > b$. So by $(\text{LS}_{a,b}^2)$ we get the pivotal equation $\gamma^{a,b} = \cdots$ for $1 \leq a < b < N$ and $a + b < p$. (8.7)

By $(\text{LS}_3^a)$, we have
\[ \gamma^{a,a} = \cdots \text{ for } 1 \leq a < N/2. \]  

(8.8)

To derive pivotal equation $\gamma^{a,a} = \cdots$ for $N/2 < a < N$ with $(N-a) \nmid N$ (thus $a \leq N-2$), we first note that, for such $a$, there must be some positive integer $k \leq N - 2$ such that
\[ 0 \leq \frac{(k-1)N}{k} < a < \frac{kN}{k+1} \leq \frac{(N-2)N}{N-1} < N - 1. \]  

(8.9)

Here $a$ is bounded with strict inequality because if $a = (k-1)N/k$ for some $k > 1$, then $N-a = N/k$ is a divisor of $N$, which is impossible by our assumption. We say such an $a$ satisfying (8.9) has height $h(a) = k$. If $h(a) = 1$, then we are in the case of (8.8). If $h(a) = 2$, then $3a < 2N$, that is, $(2a - N) + a < N$, and so, using $(\text{LS}_3^a)$, we have
\[ \gamma^{a,a} = \gamma^{2a-N,a} + \cdots = \cdots \]  

by (8.7) since $2a - N < a$. If $h(a) = 3$, then by applying $(\text{LS}_3^a)$ followed by $(\text{LS}_2^{2a-N,a})$, we get
\[ \gamma^{a,a} = \gamma^{2a-N,a} + \cdots = \gamma^{3a-2N,2a-N} + \gamma^{3a-2N,a} + \cdots = \cdots \]  

by (8.7) again since now $4a < 3N$ (and hence $5a < 4N$). Repeating this process for $a$ at higher levels, we obtain the following binary tree.
Hence, every terminal node $\gamma_{N/m}$ is a divisor of $N$. Otherwise, $N = N/(m + n)$ is a divisor of $N$, which is impossible.

(3) The weight of every node $\gamma_{ma-(m-1)N,na-(n-1)N}$ satisfies $(m + n)a - (m + n - 2)N \neq N$. Otherwise, $N - a = N/(m + n)$ is a divisor of $N$, which is impossible.

(4) Every node $\gamma_{ma-(m-1)N,na-(n-1)N}$ satisfies $ma - (m - 1)N \neq 0$. Otherwise, $N - a = N/m$ is a divisor of $N$, which is impossible.

(5) Every node $\gamma_{ma-(m-1)N,na-(n-1)N}$ satisfies $ma - (m - 1)N < na - (n - 1)N$.

Hence, every terminal node $\gamma_{i,j}$ of the tree satisfies $1 \leq i < j$ and $i + j < N$ so it can be canceled by using (8.7).

To summarize the above, we have produced the pivotal equations for the following:

(i) $\gamma_{a,b} = \cdots$ for $0 \leq a < b < N$ and $a + b < N$;
(ii) $\gamma_{a,a} = \cdots$ for $1 \leq a < N$ with $(N - a) \nmid N$.

Then we may proceed as follows:

(iii) $\gamma_{a,a} = \gamma_{a,a} + \cdots = \cdots$ for $1 \leq a < N$ by (LS$_3$) and using (ii);
(iv) $\gamma_{a,b} = \gamma_{a,b} + \gamma_{a,b} + \cdots = \cdots$ for $1 \leq a < b < N$ and $a + b = N$ by (LS$_2^{a,b}$) and (iii);
(v) $\gamma_{a,b} = \gamma_{a+b-N,a} + \gamma_{a+b-N,b} + \cdots = \cdots$ for $0 \leq a < b < N$ and $a + b > N$ by (LS$_2^{a,b}$)

and by using induction on the weight $a + b$ since the weights on the right are strictly smaller than $a + b$.

We now have completed the proof of the lemma.

Finally, Theorem 8.1 follows immediately from Propositions 8.6 and 8.7 since

$$\text{rank}(A_N) + \dim N(A_N) = N^2 + N - 1.$$

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